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ON THE

ALGEBRAICAL AND NUMERICAL

THEORY OF ERRORS OF OBSERVATIONS

AND THE

COMBINATION OF OBSERVATIONS.

ON THE

ALGEBRAICAL AND NUMERICAL THEORY OF ERRORS OF OBSERVATIONS AND THE COMBINATION OF OBSERVATIONS.

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ASTRONOMER ROYAL.

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6

PREFACE TO THE FIRST EDITION.

The Theory of Probabilities is naturally and strongly divided into two parts. One of these relates to those chances which can be altered only by the changes of entire units or integral multiples of units in the fundamental conditions of the problem; as in the instances of the number of dots exhibited by the upper surface of a die, or the numbers of black and white balls to be extracted from a bag. The other relates to those chances which have respect to insensible gradations in the value of the element measured; as in the duration of life, or in the amount of error incident to an astronomical observation.

It may be difficult to commence the investigations proper for the second division of the theory without referring to principles derived from the first. Nevertheless, it is certain that, when the elements of the second division of the theory are established, all reference to the first division is laid aside; and the original connexion is, by the great majority of persons who use the second division, entirely forgotten. The two divisions branch off into totally unconnected subjects; those persons who habitually use one part never have occasion for the other; and practically they become two different sciences.

In order to spare astronomers and observers in natural philosophy the confusion and loss of time which are produced by referring to the ordinary treatises embracing both branches of Probabilities, I have thought it desirable to draw up this tract, relating only to Errors of Observation, and to the rules, derivable from the consideration of these Errors, for the Combination of the Results of Observations. I have thus also the

advantage of entering somewhat more fully into several points, interesting to the observer, that can possibly be done in a General Theory of Probabilities.

No novelty, I believe, of fundamental character, will be found in these pages. At the same time I may state that the work has been written without reference to or distinct recollection of any other treatise (excepting only Laplace's *Théorie des Probabilités*); and the methods of treating the different problems may therefore differ in some small degrees from those commonly employed.

G. B. AIRY.

ROYAL OBSERVATORY, GREENWICH,
1861 *January* 22.

PREFACE TO THE SECOND EDITION.

The work has been thoroughly revised, but no important alteration has been made: except in the introduction of the new Section 15, and the consequent alteration in the numeration of articles of Sections 16 and 17 (formerly 15 and 16): and in the addition of the Appendix, giving the result of a comparison of the theoretical law of Frequency of Errors with the Frequency actually observed in an extensive series.

G. B. AIRY.

PREFACE TO THE THIRD EDITION.

In this Edition, Thomson and Tait's investigation of the Frequency of Errors has been substituted for Laplace's, and the investigation of Probable Errors when several quantities are concerned is introduced. No other change is made.

G. B. AIRY.

INDEX.

PART I.

FALLIBLE MEASURES AND SIMPLE ERRORS OF OBSERVATION.

SECTION 1. *Nature of the Errors here considered.*

SECTION 2. *Law of Probability of Errors of any given amount.*

SECTION 4. *Remarks on the application of these processes in particular cases.*

PART II.

ERRORS IN THE COMBINATION OF FALLIBLE MEASURES.

SECTION 5. *Law of Frequency of Error, and values of Mean Error and Probable Error, of a symbolical or numerical Multiple of one Fallible Measure.*

SECTION 6. *Law of Frequency of Error, and values of Mean Error and Probable Error, of a quantity formed by the algebraical sum or difference of two independent Fallible Measures.*

SECTION 9. *Methods of determining Mean Error and Probable Error in a given series of observations.*

PART III.

PRINCIPLES OF FORMING THE MOST ADVANTAGEOUS COMBINATION OF FALLIBLE MEASURES.

SECTION 10. *Method of combining measures; meaning of "combination-weight;" principle of most advantageous combination; caution in its application to "entangled measures."*

SECTION 14. *Instances of the formation of equations applying to several unknown quantities.*

SECTION 15. *Treatment of Observations in which it is required that the Errors of Observations rigorously satisfy some assigned conditions.*

PART IV.

ON MIXED ERRORS OF DIFFERENT CLASSES, AND CONSTANT ERRORS.

SECTION 16. *Consideration of the circumstances under which the existence of Mixed Errors of Different Classes may be recognized; and investigation of their separate values.*

CORRIGENDA.

Page 19, line 1, *for* 'Laplace's investigation, Article 14', *read* 'the two investigations Articles 12 and 13'.

Page 25, line 18, *for* 14 *read* 19.

,, 47, *dele* line 1, and *substitute* the following:—

Mean Square of Sum of Errors $a + b + c + d +$ &c.

Page 61, between lines 6 and 7, *insert* "final apparent results, as affected by the"

,, line 12, *for* 'actual error of' *read* 'apparent'.

,, line 14, *for* 'actual errors of the' *read* 'apparent'.

,, line 19, *for* 'actual error' *read* 'result'.

ON THE

ALGEBRAICAL AND NUMERICAL THEORY

OF

ERRORS OF OBSERVATIONS

AND THE

COMBINATION OF OBSERVATIONS.

PART I.

FALLIBLE MEASURES, AND SIMPLE ERRORS OF OBSERVATION.

§ 1. *Nature of the Errors here considered.*

1. THE nature of the Errors of Observation which form the subject of the following Treatise, will perhaps be understood from a comparison of the different kinds of Errors to which different Estimations or Measures are liable.

2. Suppose that a quantity of common nuts are put into a cup, and a person makes an estimate of the number. His estimate may be correct; more probably it will be incorrect. But if incorrect, the error has this

peculiarity, that it is an error of whole nuts. There cannot be an error of a fraction of a nut. This class of errors may be called Errors of Integers. These are not the errors to which this treatise applies.

3. Instead of nuts, suppose water to be put into the cup, and suppose an estimate of the quantity of water to be formed, expressed either by its cubical content, or by its weight. Either of those estimates may be in error by any amount (practically not exceeding a certain limit), proceeding by any gradations of magnitude, however minute. This class of errors may be called Graduated Errors. It is to the consideration of these errors that this treatise is directed.

4. If, instead of nuts or water, the cup be charged with particles of very small dimensions, as grains of fine sand, the state of things will be intermediate between the two considered above. Theoretically, the errors of estimation, however expressed, must be Errors of Integers of Sand-Grains; but practically, these sand-grains may be so small that it is a matter of indifference whether the gradations of error proceed by whole sand-grains or by fractions of a sand-grain. In this case, the errors are practically Graduated Errors.

5. In all these cases, the estimation is of a simple kind; but there are other cases in which the process may be either simple or complex; and, if it is complex, a different class of errors may be introduced. Suppose, for instance, it is desired to know the length of a given road.

A person accustomed to road-measures may estimate its length ; this estimation will be subject simply to Graduated Errors. Another person may measure its length by a yard-measure ; and this method of measuring, from uncertainties in the adjustments of the successive yards, &c. will also be subject to Graduated Errors. But besides this, it will be subject to the possibility of the omission of registry of entire yards, or the record of too many entire yards ; not as a fault of estimate, but as a result of mental confusion. In like manner, when a measure is made with a micrometer ; there may be inaccuracy in the observation as represented by the fractional part of the reading ; but there may also be error of the number of whole revolutions, or of the whole number of decades of subdivisions, similar to the erroneous records of yards mentioned above, arising from causes totally distinct from those which produce inaccuracy of mere observation. This class of Errors may be called Mistakes. Their distinguishing peculiarity is, that they admit of Conjectural Correction. These Mistakes are not further considered in the present treatise.

6. The errors therefore, to which the subsequent investigations apply, may be considered as characterized by the following conditions :—

They are infinitesimally graduated,
They do not admit of conjectural correction.

7. Observations or measures subject to these errors will be called in this treatise "fallible observations," or "fallible measures."

8. Strictly speaking, we ought, in the expression of our general idea, to use the word "uncertainty" instead of "error." For we cannot at any time assert positively that our estimate or measure, though fallible, is not perfectly correct; and therefore it may happen that there is no "error," in the ordinary sense of the word. And, in like manner, when from the general or abstract idea we proceed to concrete numerical evaluations, we ought, instead of "error," to say "uncertain error;" including, among the uncertainties of value, the possible case that the uncertain error may $= 0$. With this caution, however, in the interpretation of our word, the term "error" may still be used without danger of incorrectness. When the term is qualified, as "Actual Error" or "Probable Error," there is no fear of misinterpretation.

§ 2. *Law of Probability of Errors of any given amount.*

9. In estimating numerically the "probability" that the magnitude of an error will be included between two given limits, we shall adopt the same principle as in the ordinary Theory of Chances. When the numerical value of the "probability" is to be determined *à priori*, we shall consider all the possible combinations which produce error; and the fraction, whose numerator is the number of combinations producing an error which is included between the given limits, and whose denominator is the total number of possible combinations, will be the "probability" that the error will be included between those limits. But when the numerical value is to be deter-

mined from observations, then if the numerator be the number of observations, whose errors fall within the given limits, and if the denominator be the total number of observations, the fraction so formed, when the number of observations is indefinitely great, is the "probability."

10. A very slight contemplation of the nature of errors will lead us to two conclusions:—

First, that, though there is, in any given case, a possibility of errors of a large magnitude, and therefore a possibility that the magnitude of an error may fall between the two values E and $E + \delta e$, where E is large; still it is more probable that the magnitude of an error may fall between the two values e and $e + \delta e$, where e is small; δe being supposed to be the same in both. Thus, in estimating the length of a road, it is less probable that the estimator's error will fall between 100 yards and 101 yards than that it will fall between 10 yards and 11 yards. Or, if the distance is measured with a yard-measure, and mistakes are put out of consideration, it is less likely that the error will fall between 100 inches and 101 inches than that it will fall between 10 inches and 11 inches.

Second, that, according to the accuracy of the methods used and the care bestowed upon them, different values must be assumed for the errors in order to present comparable degrees of probability. Thus, in estimating the road-lengths by eye, an error amounting to 10 yards is sufficiently probable; and the chance that the real error may fall between 10 yards and 11 yards is not contemptibly

small. But in measuring by a yard-measure, the probability that the error can amount to 10 yards is so insignificant that no man will think it worth consideration; and the probability that the error may fall between 10 yards and 11 yards will never enter into our thoughts. It may, however, perhaps be judged that an error amounting to 10 inches is about as probable with this kind of measure as an error of 10 yards with eye-estimation; and the probability that the error may fall between 10 inches and 11 inches, with this mode of measuring, may be comparable with the probability of the error, in the rougher estimation, falling between 10 yards and 11 yards.

11. Here then we are led to the idea that the algebraical formula which is to express the probability that an error will fall between the limits e and $e + \delta e$ (where δe is extremely small) will possess the following properties:—

(A) Inasmuch as, by multiplying our very narrow interval of limits, we multiply our probability in the same proportion, the formula must be of the form $\phi(e) \times \delta e$.

(B) The term $\phi(e)$ must diminish as e increases, and must be indefinitely small when e is indefinitely large.

(C) The term $\phi(e)$ must contain a constant symbol or parameter c, which is constant in the expression of the probabilities under the same system of estimation or measure, and is different for different systems of estimation or measure. If (as seems likely), upon taking a proper proportion of magnitudes of error, the law of declension of the probability of errors is the same for delicate measures

and for coarse measures, then the formula will be of the form $\psi\left(\frac{e}{c}\right) \times \delta\left(\frac{e}{c}\right)$, or $\psi\left(\frac{e}{c}\right) \times \frac{\delta e}{c}$; where c is small for a delicate system of measures, and large for a coarse system of measures.

[The reader is recommended, in the first instance, to pass over the articles 12 to 21.]

12. Laplace has investigated, by an *à priori* process, well worthy of that great mathematician, the form of the function expressing the law of probability. Without entering into all details, for which we must refer to the *Théorie Analytique des Probabilités,* we may give an idea here of the principal steps of the process.

13. The fundamental principle in this investigation is, that an error, as actually occurring in observation, is not of simple origin, but is produced by the algebraical combination of a great many independent causes of error[1], each of which, according to the chance which affects it independently, may produce an error, of either sign and of different magnitude. These errors are supposed to be of the class of Errors of Integers, which admit of being treated by the usual Theory of Chances; then, supposing the integers to be indefinitely small, and the range of their number to be indefinitely great, the conditions ultimately approach to the state of Graduated Errors.

[1] This is not the language of Laplace, but it appears to be the understanding on which his investigation is most distinctly applicable to single errors of observation.

14. Suppose then that, for one source of error, the errors may be, with equal probability,

$$-n,\ -n+1,\ -n+2, \ldots -1,\ 0,\ +1,\ 2, \ldots\ n-2,\ n-1,\ n,$$

the probability of each will be $\dfrac{1}{2n+1}$.

Suppose that, for another source of error, the errors may also be, with equal probability,

$$-n,\ -n+1,\ -n+2, \ldots -1,\ 0,\ +1,\ 2, \ldots\ n-2,\ n-1,\ n,$$

and so on for s sources of error. And suppose that we wish to ascertain what is the probability that, upon combining algebraically one error taken from the first series, with one error taken from the second series, and with one error taken from the third series, and so on, we can produce an error l. The first step is, to ascertain how many are the different combinations which will each produce l.

15. Now, if we watch the process of combination, we shall see that the numbers are added by exactly the same law as the addition of indices in the successive multiplications of the polynomial

$$\epsilon^{-n\theta\sqrt{-1}} + \epsilon^{-(n-1)\theta\sqrt{-1}} + \epsilon^{-(n-2)\theta\sqrt{-1}} \ldots + \epsilon^{(n-2)\theta\sqrt{-1}} + \epsilon^{(n-1)\theta\sqrt{-1}} + \epsilon^{n\theta\sqrt{-1}},$$

by itself, supposing the operation repeated $s-1$ times. And therefore the number of combinations required will be, the coefficient of $\epsilon^{l\theta\sqrt{-1}}$ (which is also the same as the coefficient of $\epsilon^{-l\theta\sqrt{-1}}$), in the expansion of

$$\left\{\epsilon^{-n\theta\sqrt{-1}} + \epsilon^{-(n-1)\theta\sqrt{-1}} + \epsilon^{-(n-2)\theta\sqrt{-1}} \ldots + \epsilon^{(n-2)\theta\sqrt{-1}} + \epsilon^{(n-1)\theta\sqrt{-1}} + \epsilon^{n\theta\sqrt{-1}}\right\}^s.$$

This coefficient will be exhibited as a number uncombined with any power of $\epsilon^{\theta\sqrt{-1}}$, if we multiply the expansion either by $\epsilon^{l\theta\sqrt{-1}}$, or by $\epsilon^{-l\theta\sqrt{-1}}$, or by $\frac{1}{2}\left(\epsilon^{l\theta\sqrt{-1}}+\epsilon^{-l\theta\sqrt{-1}}\right)$.

The number of combinations required is therefore the same as the term independent of θ in the expansion of

$$\frac{1}{2}\left(\epsilon^{l\theta\sqrt{-1}}+\epsilon^{-l\theta\sqrt{-1}}\right)\left\{\epsilon^{-n\theta\sqrt{-1}}+\epsilon^{-(n-1)\theta\sqrt{-1}}+\&c.+\epsilon^{(n-1)\theta\sqrt{-1}}+\epsilon^{n\theta\sqrt{-1}}\right\}^s,$$

or the same as the term independent of θ in the expansion of

$$\cos l\theta \times \{1 + 2\cos\theta + 2\cos 2\theta + \ldots\ldots + 2\cos n\theta\}^s.$$

And, remarking that if we integrate this quantity with respect to θ, from $\theta = 0$ to $\theta = \pi$, the terms depending on θ will entirely disappear, and the term independent of θ will be multiplied by π, it follows that the number of combinations required is the definite integral

$$\frac{1}{\pi}\cdot\int_0^\pi d\theta\,.\cos l\theta \times \{1 + 2\cos\theta + 2\cos 2\theta \ldots + 2\cos n\theta\}^s,$$

$$\text{or } \frac{1}{\pi}\cdot\int_0^\pi d\theta\,.\cos l\theta \times \left(\frac{\sin\frac{2n+1}{2}\theta}{\sin\frac{1}{2}\theta}\right)^s.$$

And the total number of possible combinations which are, *à priori*, equally probable, is $(2n+1)^s$.

Consequently, the probability that the algebraical combination of errors, one taken from each series, will produce the error l, is

$$\frac{1}{(2n+1)^s} \cdot \frac{1}{\pi} \cdot \int_0^\pi d\theta \, . \cos l\theta \times \left(\frac{\sin \frac{2n+1}{2} \theta}{\sin \frac{1}{2} \theta} \right)^s .$$

In subsequent steps, n and s are supposed to be very large.

16. To integrate this, with the kind of approximation which is proper for the circumstances of the case, Laplace assumes

$$\frac{\sin \frac{2n+1}{2} \theta}{(2n+1) \, . \sin \frac{1}{2} \theta} = \epsilon^{-\frac{t^2}{s}};$$

(as the exponential is essentially positive, this does not in strictness apply further than $\frac{2n+1}{2} \theta = \pi$; but as succeeding values of the fraction are small, and are raised to the high power s, they may be safely neglected in comparison with the first part of the integral); expanding the sines in powers of θ, and the exponential in powers of $\frac{t^2}{s}$, it will be found that

$$\theta = \frac{t \sqrt{6}}{\sqrt{\{n(n+1)s\}}} \left(1 + \frac{B}{s} t^2 + \&c. \right),$$

where B is a function of n which approaches, as n becomes

very large, to the definite numerical value $\frac{1}{10}$. The expression to be integrated then becomes,

$$\frac{1}{\pi}\frac{\sqrt{6}}{\sqrt{\{n(n+1)s\}}}\times$$

$$\int_0^\infty dt.\cos\left[\frac{lt\sqrt{6}}{\sqrt{\{n(n+1)s\}}}\left(1+\frac{B}{s}t^2+\&c.\right)\right].\epsilon^{-t^2}.\left(1+\frac{3B}{s}t^2+\&c.\right).$$

To simplify this integral, it is to be remarked that ϵ^{-t^2} multiplies the whole, and that this factor decreases with extreme rapidity as t increases. While t is small, the terms $\frac{B}{s}t^2$ in the argument of the cosine are unimportant; and when t is large, it matters not whether they are retained or not, because their rejection merely produces a different length of period for the periodical term which is multiplied by an excessively small coefficient. Also it appears (as will be shewn in Article 19) that the integration of such a term as $\cos mt.\epsilon^{-t^2}.3Bt^2$ introduces no infinite term, and therefore when it is divided by the very large number s, this may be rejected. The integral is therefore reduced to this,

$$\frac{1}{\pi}.\frac{\sqrt{6}}{\sqrt{\{n(n+1)s\}}}\int_0^\infty dt.\cos\frac{lt\sqrt{6}}{\sqrt{\{n(n+1)s\}}}.\epsilon^{-t^2}.$$

17. As the first step to this, let us find the value of $\int_0^\infty dt.\epsilon^{-t^2}$. There is no process for this purpose so convenient as the indirect one of ascertaining the solid content of the solid of revolution in which t is the radius of any

section, and z the corresponding ordinate $= \epsilon^{-t^2}$. Let x and y be the other rectangular co-ordinates, so that $t^2 = x^2 + y^2$. Then the solid content may be expressed in either of the following ways:

By polar co-ordinates, solid content

$$= 2\pi . \int_0^\infty dt . t . \epsilon^{-t^2} = \pi.$$

By rectangular co-ordinates, solid content

$$= \int_{-\infty}^\infty dx . \int_{-\infty}^\infty dy . \epsilon^{-(x^2+y^2)} = \int_{-\infty}^\infty dx . \epsilon^{-x^2} . \int_{-\infty}^\infty dy . \epsilon^{-y^2}$$

$$= \left(4 \int_0^\infty dx . \epsilon^{-x^2}\right) \times \left(\int_0^\infty dy . \epsilon^{-y^2}\right) = 4 \left(\int_0^\infty dt . \epsilon^{-t^2}\right)^2,$$

since, for a definite integral, it is indifferent what symbol be used for the independent variable.

Hence, $$4 \left(\int_0^\infty dt . \epsilon^{-t^2}\right)^2 = \pi,$$

and $$\int_0^\infty dt . \epsilon^{-t^2} = \frac{\sqrt{\pi}}{2}.$$

18. Next, to find the value of $\int_0^\infty dt . \cos rt . \epsilon^{-t^2}$. Call this definite integral y. As this is a function of r, it can be differentiated with respect to r; and as the process of integration expressed in the symbol does not apply to r, y can be differentiated by differentiating under the integral sign. Thus

$$\frac{dy}{dr} = - \int_0^\infty dt . t \sin rt . \epsilon^{-t^2}.$$

Integrating by parts, the general integral for $\frac{dy}{dr}$

$$=\frac{1}{2}\sin rt \,.\, \epsilon^{-t^2} - \frac{r}{2}\int dt \,.\, \cos rt \,.\, \epsilon^{-t^2},$$

in which, taking the integral from $t = 0$ to $t = \infty$, the first term vanishes, and the second becomes $-\frac{r}{2}y$. Thus we have

$$\frac{dy}{dr} = -\frac{r}{2}y.$$

Integrating this differential equation in the ordinary way,

$$y = C \,.\, \epsilon^{-\frac{r^2}{4}}.$$

Now when $r = 0$, we have found by the last article that the value of y for that case is $\frac{\sqrt{\pi}}{2}$. Hence we obtain finally

$$\int_0^\infty dt \,.\, \cos rt \,.\, \epsilon^{-t^2} = \frac{\sqrt{\pi}}{2} \,.\, \epsilon^{-\frac{r^2}{4}}.$$

19. If we differentiate this expression twice with respect to r, we find,

$$\int_0^\infty dt \,.\, t^2 \,.\, \cos rt \,.\, \epsilon^{-t^2} = \sqrt{\pi} \,.\, \left(\frac{1}{4} - \frac{r^2}{8}\right)\epsilon^{-\frac{r^2}{4}};$$

and expressions of similar character if we differentiate four times, six times, &c. The right-hand expressions are never infinite. This is the theorem to which we referred in Article 16, as justifying the rejection of certain terms in the integral.

20. Reverting now to the expression at the end of Article 18, and making the proper changes of notation, we find for the value of the integral at the end of Article 16,

$$\frac{1}{2\sqrt{\pi}} \cdot \frac{\sqrt{6}}{\sqrt{\{n(n+1)s\}}} \cdot \epsilon^{\frac{-6l^2}{4n(n+1)s}}.$$

This expression for the probability that the error, produced by the combination of numerous errors (see Article 14), will be l, is based on the supposition that the changes of magnitude of l proceed by a unit at a time. If now we pass from Errors of Integers to Graduated Errors, we may consider that we have thus obtained all the probabilities that the error will lie between l and $l+1$. In order to obtain all the probabilities that the error will lie between l and $l+\delta l$, we derive the following expression from that above,

$$\frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{6}}{\sqrt{\{4n(n+1) \,.\, s\}}} \cdot \epsilon^{\frac{-6l^2}{4n(n+1) \,.\, s}} \,.\, \delta l.$$

Here l is a very large number, expressing the magnitude x of an error which is not strikingly large, by a large multiple of small units.

Let $l = mx$, where m is large; $\delta l = m\delta x$; and the probability that the error falls between x and $x + \delta x$ is

$$\frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{6} \,.\, m}{\sqrt{\{4n \,.\, (n+1) \,.\, s\}}} \cdot \epsilon^{\frac{-6m^2}{4n \,.\, (n+1) \,.\, s} x^2} \,.\, \delta x.$$

Let $\frac{4n(n+1) \,.\, s}{6m^2} = c^2$, where c may be a quantity of

magnitude comparable to the magnitudes which we shall use in applications of the symbol x; then we have finally for the probability that the error will fall between x and $x + \delta x$,

$$\frac{1}{c\sqrt{\pi}} \cdot \epsilon^{-\frac{x^2}{c^2}} \cdot \delta x.$$

This function, it will be remarked, possesses the characters which in Article 11 we have indicated as necessary. We shall hereafter call c the *modulus*.

21. Laplace afterwards proceeds to consider the effect of supposing that the probabilities of individual errors, in the different series mentioned in Article 14, are not uniform through each series, as is supposed in Article 14, but vary according to an algebraical law, giving equal probabilities for + or − errors of the same magnitude. And in this case also he finds a result of the same form. For this, however, we refer to the *Théorie Analytique des Probabilités*.

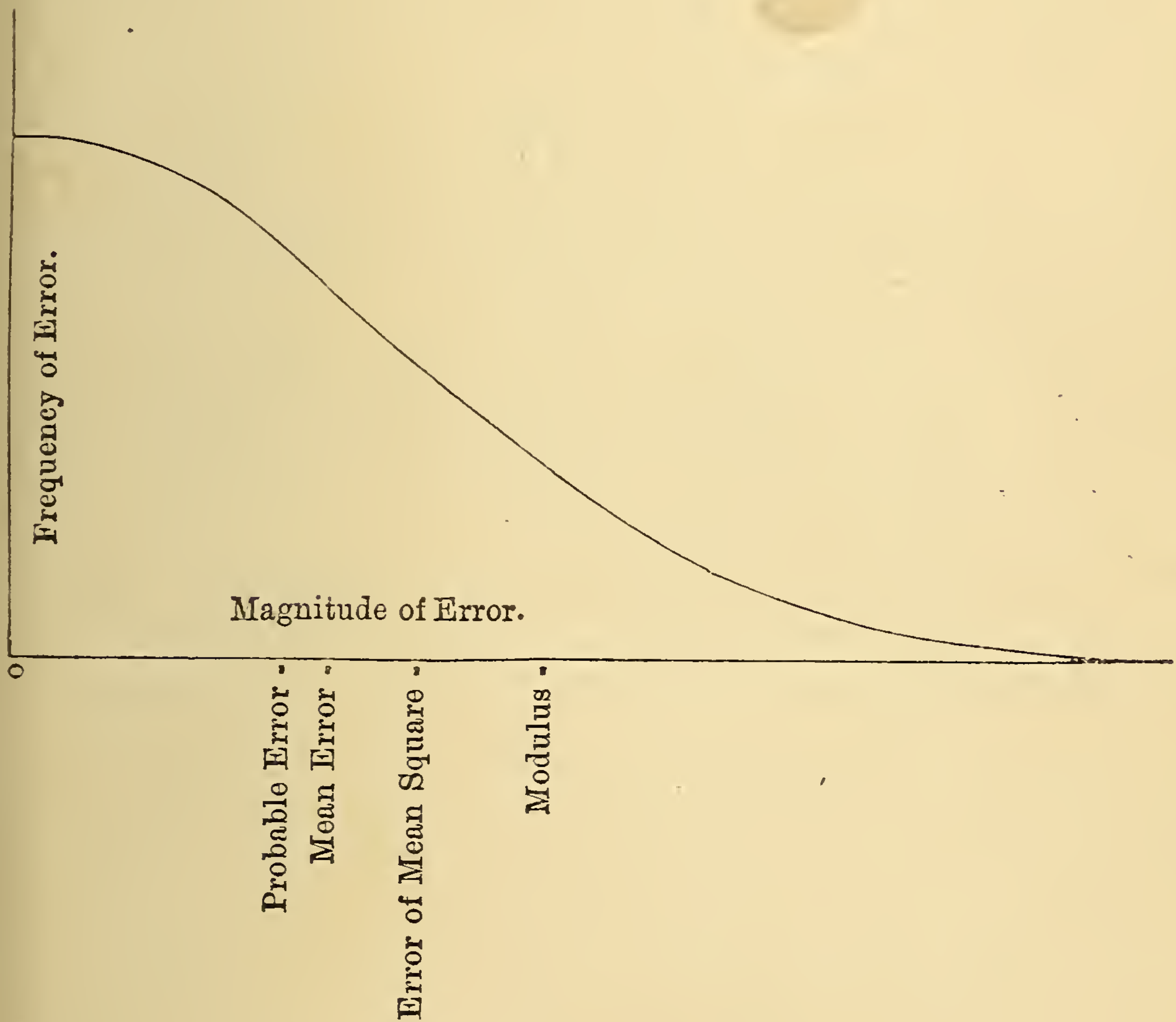
22. Whatever may be thought of the process by which this formula has been obtained, it will scarcely be doubted by any one that the result is entirely in accordance with our general ideas of the frequency of errors. In order to exhibit the numerical law of frequency (that is, the variable factor $\epsilon^{-\frac{x^2}{c^2}}$, which, when multiplied by δx, gives a number proportioned to the probability of errors falling between x and $x + \delta x$), the following table is computed;

TABLE OF VALUES OF $\epsilon^{-\frac{x^2}{c^2}}$.

$\frac{x}{c}$	$\epsilon^{-\frac{x^2}{c^2}}$	$\frac{x}{c}$	$\epsilon^{-\frac{x^2}{c^2}}$
0·0	1·0000	2·6	0·001159
0·1	0·9901	2·7	0·0006823
0·2	0·9608	2·8	0·0003937
0·3	0·9139	2·9	0·0002226
0·4	0·8521	3·0	0·0001234
0·5	0·7788	3·1	0·00006706
0·6	0·6977	3·2	0·00003571
0·7	0·6126	3·3	0·00001864
0·8	0·5273	3·4	0·000009540
0·9	0·4449	3·5	0·000004785
1·0	0·3679	3·6	0·000002353
1·1	0·2982	3·7	0·000001134
1·2	0·2369	3·8	0·0000005355
1·3	0·1845	3·9	0·0000002480
1·4	0·1409	4·0	0·0000001125
1·5	0·1054	4·1	0·00000005006
1·6	0·07731	4·2	0·00000002183
1·7	0·05558	4·3	0·000000009330
1·8	0·03916	4·4	0·000000003909
1·9	0·02705	4·5	0·000000001605
2·0	0·01832	4·6	0·0000000006461
2·1	0·01216	4·7	0·0000000002549
2·2	0·007907	4·8	0·00000000009860
2·3	0·005042	4·9	0·00000000003738
2·4	0·003151	5·0	0·00000000001389
2·5	0·001930		

23. And to present more clearly to the eye the import of these numbers, the following curve is constructed, in

which the abscissa represents $\frac{x}{c}$, or the proportion of the magnitude of an error to the modulus, and the ordinate represents the corresponding frequency of errors of that magnitude.



Here it will be remarked that the curve approaches the abscissa by an almost uniform descent from Magnitude of Error = 0 to Magnitude of Error = 1·7 × Modulus; and that after the Magnitude of Error amounts to 2·0 × Modulus, the Frequency of Error becomes practically insensible. This

is precisely the kind of law which we should *à priori* have expected the Frequency of Error to follow; and which, without such an investigation as Laplace's, we might have assumed generally; and for which, having assumed a general form, we might have searched an algebraical law. For these reasons, we shall, through the rest of this treatise, assume the law of frequency

$$\frac{1}{c\sqrt{\pi}}\epsilon^{-\frac{x^2}{c^2}}.\delta x,$$

as expressing the probability of errors occurring with magnitude included between x and $x+\delta x$.

§ 3. *Consequences of the Law of Probability or Frequency of Errors, as applied to One System of Measures of One Element.*

24. The Law of Probability of Errors or Frequency of Errors, which we have found, amounts practically to this. Suppose the total number of Measures to be A, A being a very large number; then we may expect the number of errors, whose magnitudes fall between x and $x+\delta x$, to be

$$\frac{A}{c\sqrt{\pi}}\epsilon^{-\frac{x^2}{c^2}}.\delta x,$$

where c is a modulus, constant for One System of Measures, but different for Different Systems of Measures. It is partly the object of the following investigations to give the means of determining either the modulus c, or other constants related to it, in any given system of practical errors.

25. This may be a convenient opportunity for remarking expressly that the fundamental suppositions of La-

place's investigation, Article 14, assume that the law of Probability of Errors applies equally to positive and to negative errors. It follows therefore that the formula in Article 24 must be received as applying equally to positive and to negative errors. The number A includes the whole of the measures, whether their errors may happen to be positive or negative.

26. Conceive now that the true value of the Element which is to be measured is known (we shall hereafter consider the more usual case when it is not known), and that the error of every individual measure can therefore be found. The readiest method of inferring from these a number which is closely related to the Modulus is, to take the mean of all the positive errors without sign, and to take the mean of all the negative errors without sign (which two means, when the number of observations is very great, ought not to differ sensibly), and to take the numerical mean of the two. This may be called the Mean Error. It is to be regarded as a mere numerical quantity, without sign. Its relation to the Modulus is thus found. Since the number of errors whose magnitude is included between x and $x + \delta x$ is $\frac{A}{c\sqrt{\pi}} \epsilon^{-\frac{x^2}{c^2}} . \delta x$, and the magnitude of each error does not differ sensibly from x, the sum of these errors will be sensibly $\frac{A}{c\sqrt{\pi}} \epsilon^{-\frac{x^2}{c^2}} . x\delta x$; and the sum of all the errors of positive sign will be

$$\frac{A}{c\sqrt{\pi}} \int_0^\infty dx \,.\, \epsilon^{-\frac{x^2}{c^2}} . x = \frac{cA}{2\sqrt{\pi}}.$$

The number of errors of positive sign is

$$\frac{A}{c\sqrt{\pi}}\int_0^{\infty} dx \,.\, \epsilon^{-\frac{x^2}{c^2}} = \frac{A}{2}.$$

Dividing the preceding expression by this,

$$\text{Mean positive error} = \frac{c}{\sqrt{\pi}}.$$

Similarly,

$$\text{Mean negative error} = \frac{c}{\sqrt{\pi}}.$$

And therefore,

$$\text{Mean Error} = \frac{c}{\sqrt{\pi}} = c \times 0{\cdot}564189.$$

And conversely,

$$c = \text{Mean Error} \times 1{\cdot}772454.$$

By this formula, c can be found with ease when the series of errors is exhibited.

27. It is however sometimes convenient (as will appear hereafter, Article 61) to use a method of deduction derived from the Squares of Errors. The positive and negative errors are then included under the same formula. If we form the mean of the squares, and extract the square root of that mean, we may appropriately call it the Error of Mean Square. This, like the Mean Error, is a numerical quantity, without sign. To investigate it in terms of c, we remark that the sum of the squares of errors between x and $x + \delta x$ (formed as in the last Article) will be

$$\frac{A}{c\sqrt{\pi}}\epsilon^{-\frac{x^2}{c^2}} \,.\, \delta x \times x^2$$

and the sum of all the squares of errors will be

$$\frac{A}{c\sqrt{\pi}}\int_{-\infty}^{+\infty} dx \,.\, \epsilon^{-\frac{x^2}{c^2}} \,.\, x^2 = {}_{-\infty}^{+\infty}\left\{\frac{-Ac}{2\sqrt{\pi}}\, x \,.\, \epsilon^{-\frac{x^2}{c^2}}\right\}$$

$$+\frac{Ac}{2\sqrt{\pi}}\int_{-\infty}^{+\infty} dx \,.\, \epsilon^{-\frac{x^2}{c^2}}.$$

The first term vanishes between the limits $-\infty$ and $+\infty$, and the second term $= +\frac{Ac^2}{2}$. The whole number of errors is A. Hence the Mean Square is $\frac{c^2}{2}$, and the Error of Mean Square is

$$c\sqrt{\frac{1}{2}} = c \times 0{\cdot}707107;$$

$$\text{or } c = \text{Error of Mean Square} \times 1{\cdot}414214.$$

28. It has however been customary to make use of a different number, called the Probable Error. It is not meant by this term that the number used is a more probable value of error than any other value, but that, when the positive sign is attached to it, the number of positive errors larger than that value is about as great as the number of positive errors smaller than that value: and that, when the negative sign is attached to it, the same remark applies to the negative errors. The Probable Error itself is a numerical quantity, without sign. To ascertain the algebraical condition which this requires, we have only to remark that, as the number of positive errors up to the value x is $\frac{A}{c\sqrt{\pi}}\int_0^x dx \,.\, \epsilon^{-\frac{x^2}{c^2}}$, and as the whole number of

positive errors is $\frac{A}{2}$, and half the whole number of positive errors is $\frac{A}{4}$, we must find the value of x which makes

$$\frac{1}{c\sqrt{\pi}}\int_0^x dx \,.\, \epsilon^{-\frac{x^2}{c^2}}, \text{ or } \frac{1}{\sqrt{\pi}}\int_0^w dw \,.\, \epsilon^{-w^2}, \text{ equal to } \frac{1}{4}.$$

29. For this purpose, we must be prepared with a table of the numerical values of $\frac{1}{\sqrt{\pi}}\int_0^w dw \,.\, \epsilon^{-w^2}$. It is not our business to describe here the process by which the numerical values are obtained (and which is common to the integrals of all expressible functions); we shall merely give the following table, which is abstracted from tables in Kramp's *Refractions* and in the Encyclopædia Metropolitana, Article *Theory of Probabilities*.

TABLE OF THE VALUES OF $\frac{1}{\sqrt{\pi}}\int_0^w dw \,.\, \epsilon^{-w^2}$.

w	Integral.	w	Integral.	w	Integral.
0·0	0·000000	1·1	0·440103	2·2	0·499068
0·1	0·056232	1·2	0·455157	2·3	0·499428
0·2	0·111351	1·3	0·467004	2·4	0·499655
0·3	0·164313	1·4	0·476143	2·5	0·499796
0·4	0·214196	1·5	0·483053	2·6	0·499881
0·5	0·260250	1·6	0·488174	2·7	0·499932
0·6	0·301928	1·7	0·491895	2·8	0·499962
0·7	0·338901	1·8	0·494545	2·9	0·499979
0·8	0·371051	1·9	0·496395	3·0	0·499988
0·9	0·398454	2·0	0·497661		
1·0	0·421350	2·1	0·498510	∞	0·500000

By interpolation among these, we find that the value of w which gives for the value of the integral 0·25, is 0·476948; or the Probable Error, which is the corresponding value of x, is $c \times 0{\cdot}476948$. And, conversely, $c =$ Probable Error $\times\, 2{\cdot}096665$.

30. The reader will advantageously remark in this table how nearly all the errors are included within a small value of w or $\frac{x}{c}$. For it will be remembered that the Integral when multiplied by A (the entire number of positive and negative errors) expresses the number of errors up to that value of w or $\frac{\text{error}}{c}$. Thus it appears that from $w = 0$ up to $w = 1{\cdot}65$ or $\frac{\text{error}}{c} = 1{\cdot}65$, we have already obtained $\frac{49}{50}$ of the whole number of errors of the same sign; and from $w = 0$ up to $w = 3{\cdot}0$, we have obtained $\frac{49999}{50000}$ of the whole number of errors of the same sign.

31. Returning now to the results of the investigations in Articles 26, 27, 28, 29; we may conveniently exhibit the relations between the values of the different constants therein found, by the following table:—

PROPORTIONS OF THE DIFFERENT CONSTANTS.

	Modulus.	Mean Error.	Error of Mean Square.	Probable Error.
In terms of Modulus ...	1·000000	0·564189	0·707107	0·476948
In terms of Mean Error	1·772454	1·000000	1·253314	0·845369
In terms of Error of Mean Square	1·414214	0·797885	1·000000	0·674506
In terms of Probable Error	2·096665	1·182916	1·482567	1·000000

32. To distinguish each of the errors, really occurring in observations, from the "Mean Error," "Error of Mean Square," "Probable Error," which are mere numerical deductions made according to laws framed for convenience only, we shall usually designate an error really occurring (whether its magnitude be known or not) by the term "Actual Error."

§ 4. *Remarks on the application of these processes in particular cases.*

33. It must always be borne in mind that the law of frequency of errors does not exactly hold except the number of errors is indefinitely great. With a limited number of errors, the law will be imperfectly followed; and the deductions, made on the supposition that the law is strictly followed, will be or may be inaccurate or inconsistent.

Thus, if we investigate the value of the modulus, first by means of the Mean Error, secondly by the Error of Mean Square, we shall probably obtain discordant results. We cannot assert *à priori* which of these is the better.

34. There is one case which occurs in practice so frequently that it deserves especial notice. In collecting the results of a number of observations, it will frequently be found that, while the results of the greater number of observations are very accordant, the result of some one single observation gives a discordance of large magnitude. There is, under these circumstances, a strong temptation to erase the discordant observation, as having been manifestly affected by some extraordinary cause of error. Yet a consideration of the law of Frequency of Error, as exhibited in the last Section (which recognizes the possible existence of large errors), or a consideration of the formation of a complex error by the addition of numerous simple errors, as in Article 19 14 (which permits a great number of simple errors bearing the same sign to be aggregated by addition of magnitude, and thereby to produce a large complex error), will shew that such large errors may fairly occur; and if so, they must be retained. We may perhaps think that where a cause of unfair error *may* exist (as in omission of clamping a zenith-distance-circle), and where we know by certain evidence that in some instances that unfair cause *has* actually come into play, there is sufficient reason to presume that it has come into play in an instance before us. Such an explanation, however, can only be admitted with the utmost caution.

PART II.

ERRORS IN THE COMBINATION OF FALLIBLE MEASURES.

§ 5. *Law of Frequency of Error, and values of Mean Error and Probable Error, of a symbolical or numerical Multiple of one Fallible Measure.*

35. THIS case is exceedingly simple; but it is so important that we shall make it the object of distinct treatment. Suppose that, in different measures of a quantity X, the actual errors x_1, x_2, x_3, &c. have been committed. Then it is evident that our acceptations of the value of the quantity $Y=nX$ (an algebraical or numerical multiple of X), derived from these different measures, are affected by the Actual Errors $y_1=nx_1$, $y_2=nx_2$, $y_3=nx_3$, &c.; and that, generally speaking, where X is liable to any number of errors of the magnitudes x, $x+\delta x$, or any thing between them, Y is liable to exactly the same number of errors of the magnitudes $nx=y$, $nx+n\delta x=y+\delta y$, or of magnitudes between them. Therefore the expression for the Frequency of Errors in Article 24 becomes this:

The number of errors of Y or nX, whose magnitudes fall between y and $y+\delta y$, may be expected to be

$$\frac{A}{c\sqrt{\pi}} \cdot \epsilon^{-\frac{x^2}{c^2}} \cdot \delta x,$$

which is the same as

$$\frac{A}{nc\sqrt{\pi}} \cdot \epsilon^{-\frac{y^2}{n^2c^2}} \cdot \delta y.$$

From this we at once derive these conclusions: (1) The law of Frequency of Errors for nX is exactly similar to that for the errors of an original measure X; and therefore, in all future combinations, nX may be used as if it had been an original measure. (2) The modulus for the errors of nX is, in the formula, nc. (3) Referring to the constant proportions in Articles 26, 27, 29, 31; the Mean Error of nX will therefore $= n \times$ mean error of X; the Error of Mean Square of $nX = n \times$ error of mean square of X; the Probable Error of $nX = n \times$ probable error of X.

36. It may be useful to guard the reader against one misinterpretation of the meaning of nX. We do not mean the measure of a simple quantity Y which is equal to nX. The errors (whether actual, mean, or probable) of the quantity X cannot in any way be made subservient to the determination of the error of another simple quantity Y. Thus, reverting to our instances in Article 5, &c., a judgment of the possible error in estimating the length of a road about 100 yards long will in no degree aid the judgment of the possible error in estimating the length of a road about 10000 yards long. The quantity nX is in fact merely an algebraical multiple or a numerical multiple of X, introduced into some algebraical formula, and is not exhibited as a material quantity.

37. Another caution to be observed is this; that we must most carefully distinguish between nX the multiple

of X (on the one hand), and the sum of a series of n independent quantities $X_1 + X_2 + \&c. \ldots + X_n$ (on the other hand); even though the mean error or probable error of each of the quantities X_1, X_2, &c. is equal to the mean error or probable error of X. The value of mean error or probable error of such a sum will be found hereafter (Article 53).

§ 6. *Law of Frequency of Error, and values of Mean Error and Probable Error, of a quantity formed by the algebraical sum or difference of two independent Fallible Measures.*

38. Suppose that we have the number C of measures of a quantity X, in which the law of frequency of errors is this (see Article 24), that we may expect the number of errors whose magnitudes fall between x and $x + h$, to be

$$\frac{C}{c\sqrt{\pi}} \cdot \epsilon^{-\frac{x^2}{c^2}} \cdot h;$$

c being the modulus of these errors, and the number C being very large.

And suppose that we have the number F of measures of a quantity Y, absolutely independent of the measures of the quantity X, in which we may expect the number of errors whose magnitudes fall between y and $y + h$, to be

$$\frac{F}{f\sqrt{\pi}} \cdot \epsilon^{-\frac{y^2}{f^2}} \cdot h,$$

f being the modulus of these errors, and the number F being very large.

And suppose that a new quantity Z is formed by adding X and Y together.

It is required to ascertain the law of frequency of errors of Z or $X+Y$.

39. Instead of supposing the errors to be graduated, we will suppose that all the errors whose number is $\frac{C}{c\sqrt{\pi}} . \epsilon^{-\frac{x^2}{c^2}} . h$ have the uniform magnitude x; and so for other equal intervals h in the value of x. Thus, putting $x_{\prime\prime}$, $x_{\prime}$, x', x'', for $x-2h$, $x-h$, $x+h$, $x+2h$, and putting C' for $\frac{C}{c\sqrt{\pi}}$, the numbers of errors of X will be

of magnitude $x-2h$,	$C' . \epsilon^{-\frac{x_{\prime\prime}^2}{c^2}} . h$;
of magnitude $x-h$,	$C' . \epsilon^{-\frac{x_{\prime}^2}{c^2}} . h$;
of magnitude x,	$C' . \epsilon^{-\frac{x^2}{c^2}} . h$;
of magnitude $x+h$,	$C' . \epsilon^{-\frac{x'^2}{c^2}} . h$;
of magnitude $x+2h$,	$C' . \epsilon^{-\frac{x''^2}{c^2}} . h$;

and it is plain that, by making h small enough, this state of things will approach as near as we please to that of graduated errors.

In like manner, the numbers of errors of Y will be

of magnitude $y+2h$,	$F' . \epsilon^{-\frac{y''^2}{f^2}} . h$;
of magnitude $y+h$,	$F' . \epsilon^{-\frac{y'^2}{f^2}} . h$;

of magnitude y, $\qquad F'.\epsilon^{-\frac{y^2}{f^2}}.h$;

of magnitude $y-h$, $\qquad F'.\epsilon^{-\frac{y_{\prime}^2}{f^2}}.h$;

of magnitude $y-2h$, $\qquad F'.\epsilon^{-\frac{y_{\prime\prime}^2}{f^2}}.h$.

40. Let $x+y=z$. Now in order to form all the possible values of Z, we must combine every possible value of X (C in number) with every possible value of Y (F in number), forming a total number CF of combinations. (This process implies that the errors of X are absolutely independent of those of Y.) The number of combinations of errors will be the same. If we examine the result of combining the two series of errors in the last Article, we find that the magnitudes of the errors formed will be

$z-4h$, $z-3h$, $z-2h$, $z-h$, z, $z+h$, $z+2h$, $z+3h$, $z+4h$.

We shall therefore have a series of magnitudes of error of Z in our result, varying by a step of magnitude h every time, and therefore similar to those which we have adopted for the errors of X and Y.

41. Now if we examine the combinations of errors that will produce z, and the numbers of these combinations (which apply to a step of magnitude h), we find the following:

combining $x-2h$ with $y+2h$, the number is

$$C'.\epsilon^{-\frac{x_{\prime\prime}^2}{c^2}}.h\times F'.\epsilon^{-\frac{y''^2}{f^2}}.h \text{ or } C'F'.h\times\epsilon^{-\frac{x_{\prime\prime}^2}{c^2}-\frac{y''^2}{f^2}}.h;$$

combining $x-h$ with $y+h$, the number is

$$C'F'.h \times \epsilon^{-\frac{x_{,}^2}{c^2}-\frac{y'^2}{f^2}}.h;$$

combining x with y, the number is

$$C'F'.h \times \epsilon^{-\frac{x^2}{c^2}-\frac{y^2}{f^2}}.h;$$

combining $x+h$ with $y-h$, the number is

$$C'F'.h \times \epsilon^{-\frac{x'^2}{c^2}-\frac{y_{,}^2}{f^2}}.h;$$

combining $x+2h$ with $y-2h$, the number is

$$C'F'.h \times \epsilon^{-\frac{x''^2}{c^2}-\frac{y_{,,}^2}{f^2}}.h;$$

and so on, continued indefinitely both ways. If we put $z-x$ for y, and remark that

$$y''=y+2h=z-x+2h=z-x_{,,},$$

and so for the others, we see that all the last factors in the series just exhibited are the values of

$$\epsilon^{-\frac{x^2}{c^2}-\frac{y^2}{f^2}}.h, \quad \text{or } \epsilon^{-\frac{x^2}{c^2}-\frac{(z-x)^2}{f^2}}.h,$$

when for x we put successively the values

$$x-2h,\ x-h,\ x,\ x+h,\ x+2h,$$

continued indefinitely both ways, without altering the value of z. The sum of these, supposing h made indefinitely small, is the same as

$$\int_{-\infty}^{+\infty} dx \,.\, \epsilon^{-\frac{x^2}{c^2}-\frac{(z-x)^2}{f^2}},$$

where z is considered constant. Introducing the factors, and remarking that

$$C'F'=\frac{C}{c\sqrt{\pi}} \times \frac{F}{f\sqrt{\pi}}=\frac{CF}{cf\pi},$$

the whole number of errors of magnitude z when a step of magnitude h is made each time, or, as in Article 40, the whole number of errors of Z whose magnitudes are included between z and $z+h$, will be

$$\frac{CFh}{cf\pi} \cdot \int_{-\infty}^{+\infty} dx \cdot \epsilon^{-\frac{x^2}{c^2}-\frac{(z-x)^2}{f^2}};$$

where z is to be regarded as constant.

42. The index of the exponential is easily changed into this form;

$$-\frac{z^2}{c^2+f^2}-\frac{c^2+f^2}{c^2f^2}\left(x-\frac{c^2z}{c^2+f^2}\right)^2.$$

Let $c^2+f^2=g^2$, $\dfrac{c^2+f^2}{c^2f^2}=\dfrac{1}{\gamma^2}$, $x-\dfrac{c^2z}{c^2+f^2}=\xi$.

Then the index is $-\dfrac{z^2}{g^2}-\dfrac{\xi^2}{\gamma^2}$.

And, (as $dx=d\xi$, and z is constant for this investigation), the whole number of errors of Z, whose magnitudes are included between z and $z+h$, will be

$$\frac{CFh}{cf\pi} \cdot \epsilon^{-\frac{z^2}{g^2}} \cdot \int_{-\infty}^{+\infty} d\xi \cdot \epsilon^{-\frac{\xi^2}{\gamma^2}}.$$

But $\Bigg($see Article 17, and remark that in this case

$$\left.\int_{-\infty}^{+\infty}=2\int_0^{\infty}\right),$$

$$\int_{-\infty}^{+\infty} d\xi \cdot \epsilon^{-\frac{\xi^2}{\gamma^2}}=\gamma\sqrt{\pi}=\frac{cf}{\sqrt{(c^2+f^2)}}\sqrt{\pi}=\frac{cf}{g}\sqrt{\pi}.$$

Therefore, finally, the whole number of errors of Z whose magnitudes are included between z and $z+h$, will be

$$\frac{CF}{g\sqrt{\pi}} \cdot \epsilon^{-\frac{z^2}{g^2}} \cdot h;$$

where the whole number of combinations which can form errors is CF.

43. Comparing this expression with that in Article 24, it appears that the law of frequency of error for Z is precisely the same as that for X or for Y; the modulus being g or

$$\sqrt{(c^2+f^2)}.$$

Hence we have this very remarkable result. When two fallible determinations X and Y are added algebraically to form a result Z, the law of frequency of error for Z will be the same as for X or Y, but the modulus will be formed by the theorem,

square of modulus for Z = square of modulus for X + square of modulus for Y.

44. And as (see Articles 26, 27, 28, 29, 31) the Mean Error, the Error of Mean Square, and the Probable Error, are in all cases expressed by constant numerical multiples of the Modulus, we have

$$(\text{m. e. of } Z)^2 = (\text{m. e. of } X)^2 + (\text{m. e. of } Y)^2.$$
$$(\text{e. m. s. of } Z)^2 = (\text{e. m. s. of } X)^2 + (\text{e. m. s. of } Y)^2.$$
$$(\text{p. e. of } Z)^2 = (\text{p. e. of } X)^2 + (\text{p. e. of } Y)^2.$$

These are the fundamental theorems for the Error of the Result of the Addition of Fallible Measures. They constitute, in fact, but one theorem; inasmuch as, using one, the others follow as matter of course. We shall commonly make use of Probable Errors (as most extensively adopted), unless any difference is expressly noted; but the reader, who prefers Mean Errors, may form the theorems in the corresponding shape, by merely substituting "m. e." for "p. e." throughout.

45. It cannot be too strongly enforced on the student that the measures which determine X must be absolutely and entirely independent of those which determine Y. If any one of the observations, which contributes to give a measure of X, does also contribute to give a measure of Y; then the single measure of X founded on that observation must be combined with the corresponding single measure of Y to form its value of Z, and with no other; and the freedom to combine any possible error of X with any possible error of Y, on which the whole investigation in Articles 40 and 41 depends, is to that extent lost. As an illustration: suppose that differences of astronomical latitude upon the earth, or 'amplitudes,' are determined by observations of the same stars at the two extremities of a meridian arc: and suppose that X, the amplitude from a station in the Isle of Wight to a station in Yorkshire, is determined by observing stars in the Isle of Wight and the same stars in Yorkshire; and suppose that Y, the amplitude from the Yorkshire station to a Shetland station, is determined by

observing stars in Yorkshire and the same stars in Shetland. First suppose that the observations of stars used in the measure of X are not the same which are used in the measure of Y. Then the errors in the determination of X are totally independent of the errors in the determination of Y; any one determination of X may be combined with any one determination of Y; and if $Z = X + Y =$ amplitude from Isle of Wight to Shetland, the theorem

$$(\text{p. e. of } Z)^2 = (\text{p. e. of } X)^2 + (\text{p. e. of } Y)^2$$

applies strictly. But suppose now that one and the same set of star-observations made in Yorkshire are used to determine X (by comparison with Isle of Wight observations) and Y (by comparison with Shetland observations). Then the determination of X, based upon a star-observation in Yorkshire, will be combined only with a determination of Y, based upon the same star-observation in Yorkshire (as will be seen on taking the means of zenith distances at the stations, and forming the amplitudes). The Yorkshire observations are of no use at all for determining Z, and may be completely omitted. Their errors have no influence on the result; for if the observations of any star in Yorkshire make X too small, the same observations make Y equally too large, and in forming $Z = X + Y$ these errors disappear. In fact, the determination of Z here is totally independent of those of X and Y; and the investigation of its mean error or probable error will not depend on those of X and Y. It will depend on the observations at the Isle of Wight and Shetland only: whereas

C 2

the probable error of X will depend on observations at the Isle of Wight and Yorkshire only, and the probable error of Y will depend on the observations at Yorkshire and Shetland only. Thus it may happen that, although $Z = X + Y$, the probable error of Z is less than either the probable error of X or the probable error of Y.

The investigation of the probable error of Z, when a portion of the stars observed are common to two or three stations, will be explained hereafter (Article 80).

46. Suppose that we have determinations of X and Y, as in Article 38, and $W = X - Y$; it is required to ascertain the law of frequency of errors and the mean error or probable error of W.

The fundamental supposition, upon which we have gone throughout the investigation, is, that the law of frequency is the same for positive and for negative errors of the same magnitude. And this is implied in our final formula for the number of errors between x and $x + \delta x$, namely, $\dfrac{A}{c\sqrt{\pi}} \epsilon^{-\frac{x^2}{c^2}} . \delta x$, which gives equal values for $x = +s$ and for $x = -s$. Inasmuch therefore as Y is liable to positive and negative errors of the same magnitude in equal numbers, it follows that $-Y$ is liable to the same errors as $+Y$; and therefore the probable error of $-Y$ is the same as the probable error of $+Y$.

47. Now $W = X + (-Y)$, and therefore

$$(\text{p. e. of } W)^2 = (\text{p. e. of } X)^2 + (\text{p. e. of } -Y)^2.$$

Substituting in the last term from Article 46,

$$(\text{p. e. of } W)^2 = (\text{p. e. of } X)^2 + (\text{p. e. of } Y)^2.$$

48. The theorems of Article 44 may therefore be extended, in the following form;

$$\{\text{m. e. of } (X \pm Y)\}^2 = (\text{m. e. of } X)^2 + (\text{m. e. of } Y)^2,$$

$$\{\text{e. m. s. of } (X \pm Y)\}^2 = (\text{e. m. s. of } X)^2 + (\text{e. m. s. of } Y)^2,$$

$$\{\text{p. e. of } (X \pm Y)\}^2 = (\text{p. e. of } X)^2 + (\text{p. e. of } Y)^2,$$

and the law of frequency of errors for $X \pm Y$ will be similar to that for a simple fallible measure.

49. The reader's attention is particularly invited to the following remark. We have found in Article 35 that when the errors of a fallible measure are subject to our general law of Frequency of Errors, the errors of any constant multiple of that measure are subject to the same laws; and we have found in Articles 44, 47, 48, that when the errors of each of two fallible measures are subject to that law, the errors of their sums and differences are subject to the same law. Now all our subsequent combinations of fallible quantities will consist of sums, differences, and multiples. Consequently, for every fallible quantity of which we shall treat hereafter, the General Law of Frequency of Errors will apply. Regarding this as sufficient notice, we shall not again allude to the Law of Frequency of Errors.

§ 7. *Values of Mean Error and Probable Error, in combinations which occur most frequently.*

50. To find the Probable Error of $kX + lY$, k and l being constant multipliers.

By Article 35, the probable error of $kX = k \times$ probable error of X; and the probable error of $lY = l \times$ probable error of Y. Now, considering kX and lY as two independent fallible quantities,

$$\{\text{p. e. of } (kX + lY)\}^2 = (\text{p. e. of } kX)^2 + (\text{p. e. of } lY)^2.$$

Substituting the values just found,

$$\{\text{p. e. of } (kX + lY)\}^2 = k^2 . (\text{p. e. of } X)^2 + l^2 . (\text{p. e. of } Y)^2.$$

In like manner,

$$\{\text{m. e. of } (kX + lY)\}^2 = k^2 . (\text{m. e. of } X)^2 + l^2 . (\text{m. e. of } Y)^2.$$

51. To find the Probable Error of the sum of any number of independent fallible results,

$$R + S + T + U + \&c.$$

This is easily obtained by repeated applications of the theorem of Article 44, thus:

$$\{\text{p. e. of } (R + S)\}^2 = (\text{p. e. of } R)^2 + (\text{p. e. of } S)^2;$$

$$\begin{aligned}[\text{p. e. of } \{(R + S) + T\}]^2 \\ &= \{\text{p. e. of } (R + S)\}^2 + (\text{p. e. of } T)^2 \\ &= (\text{p. e. of } R)^2 + (\text{p. e. of } S)^2 + (\text{p. e. of } T)^2;\end{aligned}$$

$$[\text{p. e. of } \{(R+S+T)+U\}]^2$$
$$= \{\text{p.e. of } (R+S+T)\}^2 + (\text{p.e. of } U)^2$$
$$= (\text{p. e. of } R)^2 + (\text{p.e. of } S)^2 + (\text{p.e. of } T)^2 + (\text{p.e. of } U)^2$$

and so on to any number of terms.

A similar theorem applies to the Error of Mean Square, and the Mean Error, substituting e.m.s. or m.e. for p.e. throughout.

52. In like manner, using the theorem of Article 50, the probable error of $rR + sS + tT + uU + \&c.$, where r, s, t, u, are constant multipliers, is given by the formula,

$$\{\text{p. e. of } (rR + sS + tT + uU)\}^2$$
$$= r^2.(\text{p. e. of } R)^2 + s^2.(\text{p. e. of } S)^2 + t^2.(\text{p.e. of } T)^2 + u^2.(\text{p. e. of } U)^2.$$

And a similar theorem for Error of Mean Square and Mean Error, substituting e.m.s. and m.e. for p.e.

Measures thus combined may be called "Cumulative Measures."

53. To find the Probable Error of $X_1 + X_2 + \ldots + X_n$, where X_1, X_2, $X_3, \ldots X_n$, are n *different and independent* measures of the same physical quantity, or of equal physical quantities, in every one of which the probable error is the same, and equal to the probable error of X_1.

By the theorem of Article 51,

$$\{\text{p.e. of } (X_1 + X_2 + \ldots + X_n)\}^2 = (\text{p.e. of } X_1)^2 + (\text{p.e. of } X_2)^2 \ldots$$
$$+ (\text{p.e. of } X_n)^2$$
$$= (\text{p.e. of } X_1)^2 + (\text{p.e. of } X_1)^2 + \ldots + (\text{p.e. of } X_1)^2 \text{ to } n \text{ terms}$$
$$= n \,.\, (\text{p. e. of } X_1)^2;$$

*

and therefore,

$$\text{p.e. of } (X_1 + X_2 + \ldots + X_n) = \sqrt{n} \times \text{p.e. of } X_1.$$

54. In Article 35, we found that

$$\text{p.e. of } nX_1 = n \times \text{p.e. of } X_1;$$

but here we find that

$$\text{p.e. of } (X_1 + X_2 \ldots + X_n) = \sqrt{n} \times \text{p.e. of } X_1,$$

although the probable error of each of the quantities X_2, X_3, &c. is equal to that of X_1. A little consideration will explain this apparent discordance. When we add together the *identical* quantities X_1, X_1, X_1, &c. to n terms; if there is a large actual error of the first X_1, there is, necessarily, the same large actual error of each of the other X_1, X_1, &c.: and the aggregate has the very large actual error $n \times$ large error of X_1. But when we add together the *independent* quantities X_1, X_2, &c., if the actual error of X_1 is large, it is very improbable that the simultaneous actual error of each of the others X_2, X_3, &c., has a value equally large and the same sign, and therefore it is very improbable that the aggregate of all will produce an actual error equal or approaching to $n \times$ large error of X_1. The magnitude of the probable error (which is proportional to the mean error, see Article 31) depends on the probability or frequency of large actual errors, (for in Article 26, to make the mean error large, we must have many large actual errors); and therefore the probable error of $X_1 + X_2 + \ldots + X_n$ will be smaller than that of $X_1 + X_1 + \ldots$ to n terms, although p. e. of $X_1 =$ p. e. of $X_2 = \ldots =$ p. e. of X_n.

55. To find the probable error of the mean of X_1, X_2, ... X_n, where $X_1, X_2, \ldots X_n$, are n different and independent measures of the same physical quantity, in every one of which the probable error = p. e. of X_1.

The mean of $X_1, X_2, \ldots X_n = \dfrac{X_1 + X_2 + \ldots + X_n}{n}$

$$= \frac{1}{n} X_1 + \frac{1}{n} X_2 + \ldots + \frac{1}{n} X_n;$$

and the square of its probable error, by Article 52,

$$= \frac{1}{n^2} (\text{p. e. of } X_1)^2 + \frac{1}{n^2} (\text{p. e. of } X_2)^2 + \ldots + \frac{1}{n^2} (\text{p. e. of } X_n)^2$$

$$= \frac{1}{n^2} (\text{p. e. of } X_1)^2 + \frac{1}{n^2} (\text{p. e. of } X_1)^2 + \ldots \text{ to } n \text{ terms}$$

$$= \frac{n}{n^2} (\text{p. e. of } X_1)^2 = \frac{1}{n} (\text{p. e. of } X_1)^2;$$

and therefore,

$$\text{p. e. of mean of } X_1, X_2, \ldots X_n = \frac{1}{\sqrt{n}} \times \text{p. e. of } X_1.$$

55*. To find the probable error of any function ϕ (W, X, Y, &c.) of one or several fallible quantities, in terms of the probable error of each. It is supposed that the values w, x, y, &c. of the fallible quantities W, X, Y, &c. are very approximately known, and therefore we may consider W as equal to $w + \delta w$, where w is an absolute constant, and δw is a very small quantity, liable to error, and where consequently error of W = error of δw, and therefore p. e. δW = p. e. δw; and so for the others. Then

$$\phi(W, X, Y, \&c.) = \phi(w, x, y, \&c.) + \frac{d}{dw} \phi(w, x, y, \&c.) \times \delta w$$

$$+ \frac{d}{dx} \phi(w, x, y, \&c.) \times \delta x + \&c.$$

where everything is constant except δw, δx, &c. Hence by the rules above,

$$\{\text{p. e. } \phi(W, X, Y, \&c.)\}^2 = \left\{\frac{d}{dw} \cdot \phi(w, x, y, \&c.)\right\}^2 \times (\text{p. e. } \delta w)^2$$

$$+ \left\{\frac{d}{dx} \cdot \phi(w, x, y, \&c.)\right\} \times (\text{p. e. } \delta x)^2$$

$$+ \text{ similar expressions for } y, z, \&c.$$

And restoring the equivalents, and remarking that, in the coefficients, w, x, y, &c. sensibly $= W$, X, Y, &c.,

$$\{\text{p. e. } \phi(W, X, Y, \&c.)\}^2 = \left\{\frac{d}{dW}\cdot\phi(W, X, Y, \&c.)\right\}^2 \times (\text{p. e. } W)^2$$
$$+\left\{\frac{d}{dX}\cdot\phi(W, X, Y, \&c.)\right\} \times (\text{p. e. } X)^2$$
$$+ \&c.$$

§ 8. *Instances of the Application of these Theorems.*

56. Instance (1). The colatitude of a geographical station is determined by observing, m times, the zenith-distance of a star at its upper culmination; and by observing, n times, the zenith-distance of the same star at its lower culmination; all proper astronomical corrections being applied. The probable error of each of the upper observations is p. e. u., and that of each of the lower is p. e. l. To find the probable error of the determination of colatitude.

The probable error of the upper zenith-distance, which is derived from the mean of m observations, is $\frac{\text{p. e. u.}}{\sqrt{m}}$; and the probable error of the lower zenith-distance, which is derived from the mean of n observations, is $\frac{\text{p. e. l.}}{\sqrt{n}}$. Now the colatitude $= \frac{1}{2}$ upper zenith-distance $+ \frac{1}{2}$ lower zenith-distance; and the determinations of these zenith-distances, as facts of observation, are strictly independent. Therefore, by Article 52,

(p. e. of colatitude)^2

$$= \tfrac{1}{4}\,(\text{p. e. of u. zen. dist.})^2 + \tfrac{1}{4}\,(\text{p. e. of l. zen. dist.})^2$$

$$= \tfrac{1}{4}\cdot\frac{(\text{p. e. u.})^2}{m} + \tfrac{1}{4}\cdot\frac{(\text{p. e. l.})^2}{n}.$$

If the observations at upper and lower culmination are equally good, so that

$$\text{p. e. u.} = \text{p. e. l.} = \text{p. e.},$$

$$\text{then (p. e. of colatitude)}^2 = \frac{(\text{p. e.})^2}{4} \cdot \left(\frac{1}{m} + \frac{1}{n}\right);$$

$$\text{or p. e. of colatitude} = \frac{\text{p. e.}}{2} \sqrt{\frac{m+n}{mn}}.$$

57. Instance (2). In the operation of determining geographical longitude by transits of the moon, the moon's right-ascension is determined by comparing a transit of the moon with the mean of the transits of several stars; to find the probable error of the right-ascension thus determined.

If p. e. m. be the probable error of moon-observation, and p. e. s. the probable error of a star-observation, and if the number of star-observations be n, then we have

$$\text{p. e. of mean of star-transits} = \frac{\text{p. e. s.}}{\sqrt{n}},$$

$$\text{p. e. of moon-transit} = \text{p. e. m.}$$

Hence, by Article 48,

p. e. of (moon-transit − mean of star-transits)

$$= \sqrt{\left\{\frac{(\text{p. e. s.})^2}{n} + (\text{p. e. m.})^2\right\}}.$$

If p. e. s. = p. e. m. = p. e.,

p. e. of (moon-transit − mean of star-transits)

$$= \text{p. e.} \sqrt{\left(\frac{1}{n} + 1\right)}.$$

It will be remarked here that, when the number of stars amounts to three or four, the probable error of result is very little diminished by increasing the number of stars.

§ 9. *Methods of determining Mean Error and Probable Error in a given series of observations.*

58. In Articles 26, 27, 28, we have given methods of determining the Mean Error, Error of Mean Square, and Probable Error, when the value of every Actual Error in a series of measures or observations is certainly known. But it is evident that this can rarely or perhaps never apply in practice, because the real value of the quantity measured is not certainly known, and therefore the value of each Actual Error is not certainly known. We shall now undertake the solution of this problem. Given a series of n measures of a physical element (all the measures being, so far as is known to the observer, equally good); to find (from the measures only) the Mean Error, Error of Mean Square, and Probable Error, of one measure, and of the mean of the n measures.

59. We shall suppose that (in conformity with a result to be found hereafter, Article 68,) the mean of the

n measures is adopted as the true result. Yet this mean is not necessarily the true result; and our investigation will naturally take the shape of ascertaining how much the formulæ of Articles 26, 27, 28, are altered by recognizing its chance of error. And first, for Mean Error. In the process of Article 26, suppose that, in consequence of our taking an erroneous value for the true result, all the + errors are increased by a small quantity, and all the − errors are diminished (numerically) by the same quantity. Then the mean + error and the mean − error will be, one increased and the other diminished, by the same quantity, and their mean, which forms the mean error, will not be affected. And if, from the same cause, one or more of the − errors become apparently + errors, the mean + error and the mean − error are very nearly equally affected in magnitude but in different ways (numerically), and their mean is sensibly unaffected. Thus the determination of Mean Error is not affected; and the process of Article 26 is to be used without alteration. A result may follow from this which is slightly inconsistent with that to be found in Article 60, as has been remarked in Article 33.

60. Secondly, for Error of Mean Square. Suppose that the Actual Errors of the n measures are a, b, c, d, &c. to n terms; then the Actual Error of the mean is

$$\frac{a+b+c+d+\&c.}{n};$$

and therefore if, for the process of Article 27, we form the sum of the squares of the Apparent Error of each mea-

sure, that is of the difference of each measure from the mean; we do not form the squares of a, b, c, d, &c., but of

$$a - \frac{a+b+c+d+\&c.}{n},$$

$$b - \frac{a+b+c+d+\&c.}{n},$$

$$c - \frac{a+b+c+d+\&c.}{n}.$$

The sum of their squares (that is, the sum of squares of apparent errors) is

$$a^2 + b^2 + c^2 + \&c.$$

$$-\frac{2}{n}(a+b+c+\&c.) \times (a+b+c+d+\&c.)$$

$$+ n \times \frac{1}{n^2} \times (a+b+c+d+\&c.)^2$$

$$= a^2 + b^2 + c^2 + \&c. - \frac{1}{n} \times (a+b+c+d+\&c.)^2.$$

Now, in the long run of observations, we may consider each of the squares in the first part of this formula as being equal to the Mean Square of Error; so that for a^2, or b^2, or c^2, &c., we may put (Error of Mean Square)^2, using the definition of Article 27. But for $a+b+c+d+\&c.$, which enters as an aggregate quantity, we must remark that, by Article 51,

Mean Square of Error of $(a+b+c+d+\&c.)$

$$= (\text{m. s. e. of } a) + (\text{m. s. e. of } b) + \&c.$$

$$= n \times (\text{Error of Mean Square})^2.$$

Thus the sum of squares which we form is truly

$$n \times (\text{e. m. s. of a measure})^2 - \frac{1}{n} \times n \times (\text{e. m. s. of a measure})^2,$$

$$= (n-1) \times (\text{e. m. s. of a measure})^2.$$

And from this,

$$\text{e. m. s. of a measure} = \sqrt{\frac{\text{sum of squares of apparent errors}}{n-1}},$$

$$\text{e. m. s. of the mean} = \sqrt{\frac{\text{sum of squares of apparent errors}}{n(n-1)}}.$$

And by the table of Article 31,

p. e. of a measure

$$= 0{\cdot}6745 \times \sqrt{\frac{\text{sum of squares of apparent errors}}{n-1}},$$

p. e. of the mean

$$= 0{\cdot}6745 \times \sqrt{\frac{\text{sum of squares of apparent errors}}{n(n-1)}}.$$

61. The quantities which enter into the formation of the mean error, error of mean square, and probable error, will be most conveniently computed thus. It is supposed that the different measures are A, B, C, &c., and that their mean is M.

First, for the mean error. Select all the measures A, B, C, &c. which are larger than M: supposing their number to be l, form the quantity

$$\frac{A+B+C+\&c.}{l}-M,$$

which gives one value of mean error. Select all the measures P, Q, R, &c., which are smaller than M; supposing their number to be s, form the quantity

$$M-\frac{P+Q+R+\&c.}{s},$$

which gives the other value of mean error. The mean of these two values of mean error is to be adopted.

Second, for the error of mean square and probable error.

We wish to form $(A-M)^2+(B-M)^2+(C-M)^2+\&c.$

This $=A^2+B^2+C^2+\&c.-2M.(A+B+C+\&c.)+n.M^2$.

$$\text{But } A+B+C+\&c.=n.M;$$

so that the expression

$$=A^2+B^2+C^2+\&c.-n.M^2.$$

This is the "Sum of squares of apparent errors," to be used in the formulæ of Article 60.

PART III.

PRINCIPLES OF FORMING THE MOST ADVANTAGEOUS COMBINATION OF FALLIBLE MEASURES.

§ 10. *Method of combining measures; meaning of "combination-weight;" principle of most advantageous combination: caution in its application to "entangled measures."*

62. THE determinations of physical elements from numerous observations, to which this treatise relates, are of two kinds.

The First is, the determination of some one physical element, which does not vary or which varies only by a certainly calculable quantity during the period of observations, by means of numerous *direct* and *immediate* measures. Thus, in the measure of the apparent angular distance between the components of a double star, we are making direct and immediate measures of a quantity sensibly invariable; in measuring the difference of moon's right ascension from the right ascension of known stars at two or more known stations, in order to render similar observations at an unknown station available for determining its longitude, we are making direct and immediate measures of quantities which are different at the two or more stations, but whose difference can be accurately computed.

63. The measures thus obtained are all fallible, and the problem before us is, How they shall be combined? It is not inconceivable that different rules might be adopted for this purpose, depending (for instance) upon the products of different powers of the various measures, and ultimate extraction of the root corresponding to the sum of the indices of powers: or upon other imaginable operations. But the one method (to which all others will approximate in effect) which has universally recommended itself, not only by its simplicity, but also by the circumstance that it permits all the measures to be increased or diminished by the same quantity (which is sometimes convenient), is, to multiply each measure by a number (either different for each different measure, or the same for any or all) which number is here called the "combination-weight;" to add together these products of measures by combination-weights; and to divide the sum by the sum of combination-weights.

64. The problem of advantageous combination now becomes this, What combination-weights will be most advantageous? And to answer this, we must decide on the criterion of advantage. The criterion on which we shall fix is:—That combination is best which gives a result whose *probable error*, or mean error, or error of mean square, is the smallest possible. This is all that we can do. We cannot assert that our result shall be correct; or that, in the case before us, its *actual error* shall be small, or smaller than might be given by many other combinations; but we can assert that it is *probable* that its actual error will be the smallest, and that it is *certain* that,

by adopting this rule in a very great number of instances, we shall on the whole obtain results which are liable to smaller errors than can be obtained in any other way.

65. Now if we know the probable errors, or the proportion of probable errors, of the individual observations, (an indispensable condition,) we can put known symbols for them, and we can put undetermined symbols for the combination-weights; and, by the precepts of Part II, we can form the symbolical expression for the probable error of the result. This probable error is to be made minimum, the undetermined quantities being the combination-weights. Thus we fall upon the theory of *complex maxima and minima.* Its application is in every case very easy, because the quantities required enter only to the second order. Instances will be found in Articles 68 to 72.

66. It sometimes happens that, even in the measures of an invariable quantity, combinations of a complicated character occur. Different complex measures are sometimes formed, leading to the same result; in which some of the observations are different in each measure, but other observations are used in all or in several of the measures; and thus the measures are not strictly independent. We shall call these "entangled measures." The only caution to be impressed on the reader is, to be very careful, in forming the separate results, to delay the exhibition of their probable errors to the last possible stage; expressing first the *actual* error of each separate result of the form ultimately required, by the *actual* error

of each observation. It will often be found that, in this way, the results of observations will be totally or partially eliminated (and justly so), which, if the probable errors had been formed at an early stage, would have vitiated the result. Instances of this will be given below (Articles 74 to 85).

67. The Second class is, the simultaneous determination of several physical elements. It may be illustrated by one of its most frequent applications, that of determining the corrections to be applied to the orbital elements of a planet's orbit. The quantities measured are right ascensions and north polar distances, observed when the planet is at different points in its orbit, and in different positions with respect to the observer. If approximate orbital elements are adopted, each having an indeterminate symbol attached to it for the small correction which it may require; it will be possible to express, by orbital calculation, every right ascension and north polar distance by numerical quantities, to which are attached definite multiples of the several indeterminate symbols. Equating these to the observed right ascensions and north polar distances, a long series of numerous equations is obtained, with different multiples of the indeterminate symbols; each equation being subject to its own *actual* error of observation. And the question before us is now, How shall these numerous equations be combined so as to form exactly as many equations as the number of indeterminate symbols, securing at the same time the condition that the probable error of every one of the values thus ob-

tained shall be the smallest possible? This is also a case of *complex maxima and minima.* Numerous problems in astronomy, geodesy, and other applied sciences, require this treatment. It will be fully explained in Articles 87 to 122.

§ 11. *Combination of simple measures; meaning of "theoretical weight;" simplicity of results for theoretical weight; allowable departure from the strict rules.*

68. Suppose that we have n independent measures of some element of observation (*e.g.* the angular distance between two stars), all equally good, so far as we can judge *à priori;* to find the proper method of combining them.

Let $E_1, E_2, \ldots E_n$, be the *actual* errors of the individual measures, which are not known, but which will affect the result. Let their probable errors be $e_1, e_2, \ldots e_n$, each of which $= e$. And let the combination-weights required be $w_1, w_2, \ldots w_n$. Then the actual error of the result, formed as is described in Article 63, will be

$$\frac{w_1E_1 + w_2E_2 \ldots + w_nE_n}{w_1 + w_2 + \ldots + w_n}$$

$$= \frac{w_1}{w_1 + w_2 \ldots + w_n} E_1 + \frac{w_2}{w_1 + w_2 \ldots + w_n} E_2 + \&c.$$

The (p. e. of result)^2, by Article 52, is

$$\left(\frac{w_1}{w_1 + w_2 \ldots + w_n}\right)^2 e_1^2 + \left(\frac{w_2}{w_1 + w_2 \ldots + w_n}\right)^2 e_2^2 + \&c.$$

$$= \frac{w_1^2 e_1^2 + w_2^2 e_2^2 \ldots + w_n^2 e_n^2}{(w_1 + w_2 \ldots + w_n)^2},$$

which in this instance becomes

$$e^2 \times \frac{w_1^2 + w_2^2 + \ldots + w_n^2}{(w_1 + w_2 \ldots + w_n)^2}.$$

Making the fraction minimum with respect to w_1, we obtain

$$\frac{2w_1}{w_1^2 + w_2^2 \ldots + w_n^2} - \frac{2}{w_1 + w_2 \ldots + w_n} = 0.$$

Similarly, by w_2,

$$\frac{2w_2}{w_1^2 + w_2^2 \ldots + w_n^2} - \frac{2}{w_1 + w_2 \ldots + w_n} = 0,$$

and so for the other weights.

It follows that $w_1 = w_2 = w_3$, &c., but that all are indeterminate. That is, the measures are to be combined by equal combination-weights; or, in other words, the arithmetical mean is to be taken. The (probable error of result)^2 $= \frac{e^2}{n}$; or,

$$\text{probable error of result} = \frac{e}{\sqrt{n}};$$

as was found in Article 55.

69. Suppose that we have n independent measures or results which are not equally good. (For instance: the atmospheric or other circumstances of individual observations may be different: or, if individual observations are equally good, the results of different days, formed by the means of different numbers of observations on the different days, would have different values. In determinations of colatitude by means of different stars, the values of results from different stars will be affected by their north polar distances, as well as by the other circumstances.)

The notations of Article 68 may be retained, rejecting only the simple letter e. Thus we have for (p. e. of result)^2,

$$\frac{w_1^2 e_1^2 + w_2^2 e_2^2 \ldots + w_n^2 e_n^2}{(w_1 + w_2 \ldots + w_n)^2};$$

and w_1, w_2, &c., are to be so determined as to make this minimum.

Differentiating with respect to w_1,

$$\frac{2w_1e_1^2}{w_1^2e_1^2 + w_2^2e_2^2 \ldots + w_n^2e_n^2} - \frac{2}{w_1 + w_2 \ldots + w_n} = 0.$$

Differentiating with respect to w_2,

$$\frac{2w_2e_2^2}{w_1^2e_1^2 + w_2^2e_2^2 \ldots + w_n^2e_n^2} - \frac{2}{w_1 + w_2 \ldots + w_n} = 0.$$

And so for the others.

It is evident that $w_1e_1^2 = w_2e_2^2 = \&c. = w_ne_n^2 = C$ some indeterminate constant. Hence

$$w_1 = \frac{C}{e_1^2}, \quad w_2 = \frac{C}{e_2^2}, \quad \&c., \quad w_n = \frac{C}{e_n^2},$$

and (p.e. of result)^2

$$= \frac{C(w_1 + w_2 \ldots + w_n)}{(w_1 + w_2 \ldots + w_n)^2} = \frac{C}{w_1 + w_2 \ldots + w_n}.$$

Or $$\frac{1}{(\text{p. e. of result})^2} = \frac{1}{e_1^2} + \frac{1}{e_2^2} + \ldots + \frac{1}{e_n^2}.$$

70. We shall now introduce a new term. Let

$$\frac{1}{(\text{probable error})^2}$$

be called the "theoretical weight," or t. w. Then we have these two remarkable results:—

When independent fallible measures are *collateral*, that is, when each of them gives a measure of the same unknown quantity, which measures are to be combined by combination-weights in order to obtain a final result;—

First. The *combination-weight* for each measure ought to be proportional to its *theoretical weight.*

Second. When the combination-weight for each measure is proportional to its theoretical weight, the *theoretical weight* of the final result is equal to the *sum of the theoretical weights* of the several *collateral* measures[1].

When the theoretical weights of the original fallible measures are equal, and they are combined with equal combination-weights, the theoretical weight of the result is proportional to the number of the original measures.

71. These rules apply in every case of combination of measures leading to the value of the same simple quantity, provided that the observations on which those measures are founded are absolutely independent. Thus, we may combine by these rules the measures of distance or position of double stars made on different days; the zenith distances of the same star (for geographical latitude) on different days; the results (for geographical latitude) of the observations of different stars; the results (for geodetic amplitude) of the observations of different stars; the results (for terrestrial longitudes) of transits of the moon on different days, &c.

72. Instance. In Article 56 we have found for the probable error of colatitude determined by m observations

[1] The reader is cautioned, while remembering these important theorems, also to bear in mind the following (Articles 44 to 52):—

When independent fallible measures or quantities are *cumulative*, that is, when they are to be combined by addition or subtraction to form a new fallible quantity; then the *square of probable error* of the new fallible quantity is equal to the *sum of the squares of probable errors* of the several *cumulative* measures or quantities.

of a star at its upper culmination, and n observations at its lower culmination,

$$\frac{e}{2}\sqrt{\frac{m+n}{mn}},$$

where e is the probable error of an observation, all being supposed equally good. Another star, whose observations are equally good, observed m_1 times at upper and n_1 times at lower culmination, gives a result with probable error

$$\frac{e}{2}\sqrt{\frac{m_1+n_1}{m_1n_1}},$$

a third gives a result with probable error

$$\frac{e}{2}\sqrt{\frac{m_2+n_2}{m_2n_2}}, \text{ \&c.}$$

Their theoretical weights are

$$\frac{4}{e^2}\cdot\frac{mn}{m+n}, \quad \frac{4}{e^2}\cdot\frac{m_1n_1}{m_1+n_1}, \quad \frac{4}{e^2}\cdot\frac{m_2n_2}{m_2+n_2}, \quad \text{\&c.}$$

The different results ought to be combined (to form a mean) with combination-weights proportional respectively to

$$\frac{mn}{m+n}, \quad \frac{m_1n_1}{m_1+n_1}, \quad \frac{m_2n_2}{m_2+n_2}, \text{ \&c.};$$

and the theoretical weight of the mean so formed will be

$$\frac{4}{e^2}\left(\frac{mn}{m+n}+\frac{m_1n_1}{m_1+n_1}+\frac{m_2n_2}{m_2+n_2}+\text{\&c.}\right),$$

and its probable error will be the square root of the reciprocal of this quantity.

It is supposed here that the zenith-point is free from error. If it is not, the case becomes one of "entangled observations," similar to that of Article 75.

73. We may however depart somewhat from the precise rule of combination laid down in Article 70, without materially vitiating our results. We have in Article 69 determined the conditions which make p. e. of result minimum; and it is well known that, in all cases of algebraical minimum, the primary variable may be altered through a considerable range, without giving a value of the derived function much differing from the minimum. Thus, suppose that we had two independent measures, for the same physical element, whose probable errors were e and $e'=2e$. We ought, by the rule of Article 70, to combine them by combination-weights in the proportion of 4 : 1. But suppose that we use combination-weights in the proportion of $n : 1$. Put E and E' for the actual errors; the actual error of result will be

$$\frac{nE+E'}{n+1}=\frac{n}{n+1}E+\frac{1}{n+1}E';$$

the p. e. will be (by Article 52)

$$\sqrt{\left\{\left(\frac{n}{n+1}\right)^2 . e^2+\left(\frac{1}{n+1}\right)^2 . e'^2\right\}}=e . \frac{\sqrt{(n^2+4)}}{n+1}$$

$$=e\sqrt{\left\{1+\frac{3-2n}{(n+1)^2}\right\}}.$$

Using special numbers, we find

With combination-weights as 2 : 1, the p. e. of result

$$=e\frac{\sqrt{8}}{3}=e\times 0{\cdot}943.$$

With combination-weights as 4 : 1, the p. e. of result

$$= e\frac{\sqrt{20}}{5} = e \times 0{\cdot}894.$$

.... 8 : 1, the p. e. of result

$$= e\frac{\sqrt{68}}{9} = e \times 0{\cdot}916.$$

.... 16 : 1, the p. e. of result

$$= e\frac{\sqrt{260}}{17} = e \times 0{\cdot}947.$$

Thus it appears that we may use combination-weights in any proportion between those of 2 : 1 and 16 : 1 without increasing the p. e. of result by more than $\frac{1}{15}$ part.

But if we used a proportion of combination-weights less than 3 : 2, the probable error of the result would be greater, and the value of the result less, than if we used the principal measure alone.

The values of the result obtained by these combinations will be different, but we have no means of knowing with certainty whether one approaches nearer to the truth than another. All that we know is that, in repeating combinations of these kinds in an infinite number of instances, that which we have indicated as best will on the whole produce rather smaller errors than the others.

When, however, we depart from the strictness of the First rule in Article 70, the Second theorem of that Article no longer holds.

§ 12. *Treatment of Entangled Measures.*

74. The nature and treatment of entangled measures will be best understood from instances.

Instance (1). Suppose that the longitude of an unknown station is to be determined by the right ascension of the moon at transit (as found by ascertaining the difference between the moon's time of transit and the mean of the times of transit of n stars) compared with the right ascension at transit determined in the same manner at a known station (where the number of stars observed is a); and suppose the probable error of transit of the moon or of any star to be e. Then, as has been found in Article 57, the probable error of right ascension at the unknown station is $e\sqrt{\left(\frac{1}{n}+1\right)}$, that at the known station is $e\sqrt{\left(\frac{1}{a}+1\right)}$; and therefore, by Articles 47 and 48, as these two determinations are in every respect independent, the probable error of the difference of right ascensions at transit (on which the longitude depends) is $e\sqrt{\left(\frac{1}{n}+\frac{1}{a}+2\right)}$.

Suppose that a second comparison is made, of the same transits at the unknown station, with transits of the moon and b stars at a second known station. The probable error of the quantity on which the longitude depends is found in like manner to be $e\sqrt{\left(\frac{1}{n}+\frac{1}{b}+2\right)}$.

Now if we combined these two results, (leading to the same physical determination, and both correct,) by the rules of Article 70, we should obtain an erroneous conclusion. For, the two results are not independent, inasmuch as the observations at the unknown station enter into both.

75. To obtain a correct result, we must refer to the *actual* errors. In strictness, we ought to refer to the actual error of each individual observation; but, inasmuch as it is perfectly certain that all the observations at each of the stations, separately considered, are entirely independent of all the observations at the other stations, we may put a symbol for the *actual* error of excess of moon's R.A. above mean of stars' R.A. at each of the stations. Let these symbols be N, A, B, respectively. Then the *actual* errors of the quantities on which longitude depends, as found by comparing the unknown station with each of the known stations, are respectively $N-A$, $N-B$. Let the quantities be combined with the combination-weights α, β. Then the final actual error will be

$$\frac{\alpha(N-A)+\beta(N-B)}{\alpha+\beta}=N-\frac{\alpha}{\alpha+\beta}A-\frac{\beta}{\alpha+\beta}B.$$

And the square of probable error of final result

$$=\left\{(\text{p.e. of } N)^2+\frac{\alpha^2}{(\alpha+\beta)^2}(\text{p.e. of } A)^2+\frac{\beta^2}{(\alpha+\beta)^2}(\text{p. e. of } B)^2\right\}.$$

To make this minimum, we must make

$$\frac{\alpha^2(\text{p. e. of } A)^2+\beta^2(\text{p. e. of } B)^2}{(\alpha+\beta)^2}$$

minimum. This algebraical problem is exactly the same as that of Article 69, and the result is

$$\alpha = \frac{C}{(\text{p.e. of } A)^2}, \quad \beta = \frac{C}{(\text{p.e. of } B)^2},$$

where C is an indeterminate constant. And this gives for (p.e. of final result)^2,

$$\left\{(\text{p. e. of } N)^2 + \frac{C}{\alpha+\beta}\right\}$$

$$= \left\{(\text{p.e. of } N)^2 + \frac{(\text{p.e. of } A)^2 \times (\text{p.e. of } B)^2}{(\text{p.e. of } A)^2 + (\text{p.e. of } B)^2}\right\}$$

$$= e^2 \left\{\frac{1}{n} + 1 + \frac{(1+a)(1+b)}{a+b+2ab}\right\}^* .$$

76. If we put r for the "theoretical weight" of final result (see the definition in Article 70); n, a, b, for those of the observations N, A, B, respectively; then the last formula but one becomes

$$\frac{1}{\mathrm{r}} = \frac{1}{\mathrm{n}} + \frac{1}{\mathrm{a}+\mathrm{b}};$$

or

$$\mathrm{r} = \frac{(\mathrm{a}+\mathrm{b})\,\mathrm{n}}{\mathrm{n}+(\mathrm{a}+\mathrm{b})}.$$

Let n be divided into two parts $\mathrm{n_a}$ and $\mathrm{n_b}$, such that

$$\mathrm{n_a} = \frac{\mathrm{a}}{\mathrm{a}+\mathrm{b}}\,\mathrm{n}, \quad \mathrm{n_b} = \frac{\mathrm{b}}{\mathrm{a}+\mathrm{b}}\,\mathrm{n}.$$

Now if the theoretical weight $\mathrm{n_a}$ at the station N had

* Instances of a more complicated character may be seen in the Memoirs of the R. Astronomical Society, Vol. XIX. p. 213.

been combined with the theoretical weight a at the station *A*, they would have given for theoretical weight of their result

$$r_a = \frac{a \,.\, n_a}{n_a + a} = \frac{\dfrac{a^2 n}{a+b}}{\dfrac{an}{a+b} + a} = \frac{an}{n + (a+b)}.$$

And if the theoretical weight n_b at the station *N* had been combined with the theoretical weight b at the station *B*, they would have given for theoretical weight of their result

$$r_b = \frac{b \,.\, n_b}{n_b + b} = \frac{\dfrac{b^2 n}{a+b}}{\dfrac{bn}{a+b} + b} = \frac{bn}{n + (a+b)}.$$

And consequently,

$$r_a + r_b = r.$$

And it is easy to see that, as there are two conditions to be satisfied by the two quantities n_a, n_b, no other quantities will produce the same aggregates n and r.

77. Hence we may conceive that the theoretical weight n is divided into two parts proportional to a and b, and that those parts are combined separately with a and b respectively, and that they produce in the result the separate parts r_a and r_b, which united make up the entire theoretical weight of result r. The same, it would be found, applies if there are any number of stations *A*, *B*, *C*, *D*, &c.

78. The partition of theoretical weight of final result thus obtained, producing separate theoretical weights of result depending on the combination of N with A and N with B respectively, does in fact produce separate theoretical weights for comparison of N with A, and comparison of N with B, without necessarily distinguishing whether the element (as moon's place) to which N relates is inferred from that to which A relates, or whether the element to which A relates is inferred from that to which N relates. Hence it is applicable to such cases as the following.

79. Instance (2). A geodetic theodolite being considered immoveable, observations (whose actual error is M) are made with it for the direction of the north meridian, and observations (subject to actual errors A, B, C, &c.) are made on different triangulation-signals: to find the weight to be given to the determination of the true azimuth of each signal.

Using analogous notation, the theoretical weight m is to be divided into parts m_a, m_b, m_c, &c.; and then the weights of the determinations for separate signals are those produced by combining m_a with a, m_b with b, &c., or are

$$\frac{am}{m + (a + b + c \ \&c.)}, \quad \frac{bm}{m + (a + b + c \ \&c.)}, \ \&c.$$

80. Instance (3). In the observation of zenith-distances of stars for the amplitude of a meridian arc divided

into two sections by an intermediate station: suppose that a stars are observed at all the stations, the means of actual errors being respectively A_1, A_2, A_3: suppose that b stars are observed at the first and second stations only, the means of the actual errors being respectively B_1, B_2: that c stars are observed at the second and third only, the means of actual errors being C_2, C_3: and that d stars are observed at the first and third only, the means of actual errors being D_1, D_3. They may be represented thus:

	Stars observed at first station.	Stars observed at second station.	Stars observed at third station.
Stars a	A_1,	A_2,	A_3.
Stars b	B_1,	B_2.	
Stars c		C_2,	C_3.
Stars d	D_1,		D_3.

Suppose the probable error of every individual observation to be e. It is now required to find the combination proper for determining the amplitude of the first section of the arc.

81. Besides the direct measures of the first section, there are indirect measures produced by subtracting the measures of the second section from the measures of the

whole arc. All the possible measures of the first section are therefore the following:

$$\left.\begin{array}{rl} \text{I.} & A_2 - A_1 \\ \text{II.} & B_2 - B_1 \end{array}\right\} \text{Direct.}$$

$$\left.\begin{array}{rl} \text{III.} & A_3 - A_1 - (A_3 - A_2) \\ \text{IV.} & A_3 - A_1 - (C_3 - C_2) \\ \text{V.} & D_3 - D_1 - (A_3 - A_2) \\ \text{VI.} & D_3 - D_1 - (C_3 - C_2) \end{array}\right\} \text{Indirect.}$$

But of these, III is a mere reproduction of I: and of the four measures I, IV, V, VI, it is easily seen that one may be formed from the three others; and the retention of all would introduce indeterminate solutions. The following may be retained, as substantially different;

$A_2 - A_1$................ with combination-weight v,

$B_2 - B_1$.. w,

$A_3 - A_1 - C_3 + C_2$.................................... x,

$D_3 - D_1 - C_3 + C_2$.................................... y.

These are entangled measures, inasmuch as A_1, C_2, C_3, appear in different measures.

82. The actual error of their mean will be

$$\frac{v(A_2 - A_1) + w(B_2 - B_1) + x(A_3 - A_1 - C_3 + C_2) + y(D_3 - D_1 - C_3 + C}{v + w + x + y}$$

$$= \frac{-(v+x)A_1 + vA_2 + xA_3 - wB_1 + wB_2 + (x+y)C_2 - (x+y)C_3 - yD_1 + yD}{v + w + x + y}$$

The independent fallible quantities are now separated; and, by Article 52, remarking that $(\text{p. e. of } A_1)^2 = \frac{e^2}{a}$, and so for the others, we find

$$\left.\frac{\text{e. of result}}{e}\right)^2$$

$$\frac{v+x)^2\frac{1}{a}+v^2\frac{1}{a}+x^2\frac{1}{a}+w^2\frac{1}{b}+w^2\frac{1}{b}+(x+y)^2\frac{1}{c}+(x+y)^2\frac{1}{c}+y^2\frac{1}{d}+y^2\frac{1}{d}}{(v+w+x+y)^2}.$$

Making this minimum with regard to v, w, x, y, as in Article 69,

$$(v+x)\frac{1}{a}+v\frac{1}{a}=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad C,$$

$$w\frac{1}{b}+w\frac{1}{b}=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad C,$$

$$(v+x)\frac{1}{a}+x\frac{1}{a}+(x+y)\frac{1}{c}+(x+y)\frac{1}{c}=C,$$

$$(x+y)\frac{1}{c}+(x+y)\frac{1}{c}+y\frac{1}{d}+y\frac{1}{d}\ldots\ldots=C.$$

From which

$$v=\frac{4a^2+2ac+4ad}{8a+6c+6d}\,C,$$

$$w=\frac{b}{2}\,C=\frac{4ab+3bc+3bd}{8a+6c+6d}\,C,$$

$$x=\frac{2ac-2ad}{8a+6c+6d}\,C,$$

$$y=\frac{2ad+3cd}{8a+6c+6d}\,C.$$

It is remarkable here that in some cases x may be negative, indicating that advantage will be gained by subtracting that multiple of measure from the others.

If $a=b=c=d$, and $D=aC$, the combination-weights become

$$v=\frac{D}{2},\quad w=\frac{D}{2},\quad x=0,\quad y=\frac{D}{4}.$$

83. If we thought fit to reject the combination

$$A_3-A_1-C_3+C_2,$$

there would be no entanglement; and it would easily be found, by Article 70, that the combination-weights ought to be proportional to a, b, $\frac{cd}{c+d}$; and the theoretical weight of the result

$$=\frac{1}{e^2}\left(\frac{a}{2}+\frac{b}{2}+\frac{cd}{2c+2d}\right).$$

In like manner, for the second section of the arc, the measures to be used are

$$A_3-A_2,\quad C_3-C_2,\quad D_3-D_1-B_2+B_1;$$

and the theoretical weight of result

$$=\frac{1}{e^2}\left(\frac{a}{2}+\frac{c}{2}+\frac{bd}{2b+2d}\right).$$

84. Now if we combined these two sections to form the whole arc, and inferred the probable error of the whole from the probable errors of the sections by the rule of

Article 44, we should obtain an erroneous result. For, the observations on which the determinations of value of the two sections are founded are not independent; both contain the observations A_2, B_1, B_2, C_2, C_3, D_1, D_3; and they are therefore entangled results.

The correct result for the whole will be obtained by an investigation exactly similar to that for each part. There is the direct measure by the a stars, with error $A_3 - A_1$; the direct measure by the d stars, with error $D_3 - D_1$; and the indirect measure obtained by adding the result of the b stars to the result of the c stars, with error $B_2 - B_1 + C_3 - C_2$. The theoretical weight of the result will be found to be

$$\frac{1}{e^2}\left(\frac{a}{2}+\frac{d}{2}+\frac{bc}{2b+2c}\right).$$

If the number of observations at the intermediate station is very small, (as if a is small, b and $c = 0$, d large,) the theoretical weight of the value of each section will be small, while that of the entire arc may be great.

This instance is well adapted to give the reader a clear idea of the characteristic difference between *actual* error and *probable* error. So far as *actual* error is concerned, if we add the measure of one section with its *actual* error, to the measure of the other section with its *actual* error, we entirely (and correctly) destroy so much of the *actual* error as depends on the observations at the intermediate station. But the *probable* error (see Article 8) is a mea-

sure of *uncertainty;* and if, without looking carefully in each case to the origin of the uncertainty, we simply add together the two separate measures charged with their respective *uncertainties,* we obtain for the whole arc a sum with an apparently large *uncertainty* which is very incorrect.

85. If the observations at the three stations are to be combined in one connected system; it will be best to use each batch of stars separately, giving to each resulting amplitude its proper weight as deduced from that batch only. For the batches B, C, D, the operation is perfectly clear; for the batch A, the principles of Articles 75, 76, 79, must be used, which here give a very simple result.

86. It is scarcely necessary to delay longer on the subject of entangled measures. The caution required, and which in all cases suffices, is:—to commence the investigations by the use not of probable but of actual errors; to collect all the coefficients of each actual error, and to separate them from the coefficients of every other error; and when the formulæ are in a state fit for the introduction of probable errors, to investigate, by a process special to the case under consideration, the magnitudes of the combination-weights which will produce the minimum probable error in the result.

§ 13. *Treatment of numerous equations applying to several unknown quantities: introduction of the term "minimum squares."*

87. The origin of equations of this class has been explained in Article 67. It has there been seen that, putting x, y, &c., for the corrections to orbital elements, &c. which it is the object of the problem to discover, (the number of which elements we shall for clearness suppose to be three, though the investigation will evidently apply in the same form to any number of such corrections,) every equation will have the form

$$ax + by + cz = f,$$

where f is the difference between a quantity computed theoretically from assumed elements and a quantity observed, and is therefore subject to the casual error of observation. If the last terms of the equations, as given immediately by observation, have not the same probable error, we shall suppose that the equations are multiplied by proper factors (see Article 35), so that in every case the probable error of the last term f is made $= e$; e being an arbitrary number, for which sometimes it is very convenient to substitute the abstract value 1. We shall use the letters a, b, c, f, and others which are to be introduced, without subscripts, in their general sense; but for the separate equations we shall affix the subscripts 1, 2, &c.

88. The number of equations being greater than three, and it being requisite to reduce the final equations to three in number; the only method which suggests itself, for giving every one of the fundamental equations a proper share in the formation of those three equations, is:—first to multiply the equations by a series of factors k_1, k_2, &c., and to adopt their sum as one fundamental equation; secondly, to multiply them by another series l_1, l_2, &c.; thirdly, to multiply them by another series m_1, m_2, &c. Thus having the series of fundamental equations

$$a_1x + b_1y + c_1z = f_1,$$

$$a_2x + b_2y + c_2z = f_2,$$

&c.

we form the three series

$$k_1a_1x + k_1b_1y + k_1c_1z = k_1f_1,$$

$$k_2a_2x + k_2b_2y + k_2c_2z = k_2f_2,$$

&c.

$$l_1a_1x + l_1b_1y + l_1c_1z = l_1f_1,$$

$$l_2a_2x + l_2b_2y + l_2c_2z = l_2f_2,$$

&c.

$$m_1a_1x + m_1b_1y + m_1c_1z = m_1f_1,$$

$$m_2a_2x + m_2b_2y + m_2c_2z = m_2f_2,$$

&c.

of which the sums are

$$x \, . \, \Sigma \, (ka) + y \, . \, \Sigma \, (kb) + z \, . \, \Sigma \, (kc) = \Sigma \, (kf),$$
$$x \, . \, \Sigma \, (la) + y \, . \, \Sigma \, (lb) + z \, . \, \Sigma \, (lc) = \Sigma \, (lf),$$
$$x \, . \, \Sigma \, (ma) + y \, . \, \Sigma \, (mb) + z \, . \, \Sigma \, (mc) = \Sigma \, (mf).$$

These are our three final equations for determining x, y, and z: and our problem now is, to ascertain the law of formation of the factors k, l, m, which will give values of x, y, z, for each of which the probable error may be minimum.

89. Let us confine our attention, for a short time, to the investigation of the value of x. The process of solving the last three equations will consist, in fact, in finding different factors wherewith the equations may be multiplied, such that, when the multiplied equations are added together, y and z may be eliminated, and the terms depending on x and f may alone remain. But, remarking how the three equations are composed from the original equations, this multiplication of equations formed by sums of multiples of the original equations is in fact a collection of sums of other multiples of the original equations. Let n be the general letter for the multipliers (formed by this double process) of the original equations; then the final process for solution of the equations is thus exhibited;

$$x \times \Sigma \, (na) = \Sigma \, (nf) \, ;$$
$$\Sigma \, (nb) = 0 \, ;$$
$$\Sigma \, (nc) = 0 \, ;$$

which can be solved with an infinity of different values of n.

90. From these,

$$x = \frac{\Sigma(nf)}{\Sigma(na)} = \frac{n_1 f_1 + n_2 f_2 + \&c.}{n_1 a_1 + n_2 a_2 + \&c.};$$

from which the actual error of x

$$= \frac{n_1}{n_1 a_1 + n_2 a_2 + \&c.} \times \text{actual error of } f_1$$

$$+ \frac{n_2}{n_1 a_1 + n_2 a_2 + \&c.} \times \text{actual error of } f_2$$

$$+ \&c.,$$

and, as the probable error of each of the quantities f_1, f_2, &c. $= e$, the square of probable error of x

$$= e^2 \times \frac{n_1^{\ 2} + n_2^{\ 2} + \&c.}{(n_1 a_1 + n_2 a_2 + \&c.)^2}$$

$$= e^2 \times \frac{\Sigma(n^2)}{\{\Sigma(na)\}^2}.$$

The numbers n_1, n_2, &c. are so to be chosen that the square of probable error of x shall be minimum; and therefore its variation produced by simultaneous small variations in each of them shall be 0.

If we put δn_1, δn_2, &c. for such small variations, we must have, by the formulæ of ordinary differentiation,

$$0 = \frac{n_1\delta n_1 + n_2\delta n_2 + \&c.}{n_1^2 + n_2^2 + \&c.} - \frac{a_1\delta n_1 + a_2\delta n_2 + \&c.}{n_1a_1 + n_2a_2 + \&c.};$$

or $$0 = \frac{n_1\delta n_1 + n_2\delta n_2 + \&c.}{\Sigma(n^2)} - \frac{a_1\delta n_1 + a_2\delta n_2 + \&c.}{\Sigma(na)} \ldots\ldots [1].$$

But the variations δn_1, δn_2, &c., are not independent here, as were the corresponding variations in Articles 68 and 69; for they are affected by the antecedent conditions $\Sigma(nb) = 0$, $\Sigma(nc) = 0$; from which we derive

$$0 = b_1\delta n_1 + b_2\delta n_2 + \&c. \ldots\ldots\ldots\ldots\ldots [2],$$

$$0 = c_1\delta n_1 + c_2\delta n_2 + \&c. \ldots\ldots\ldots\ldots\ldots [3].$$

These three equations must hold simultaneously for the values of n_1, n_2, &c., which we require.

91. It would perhaps be a troublesome matter to extract *analytically* from these equations the values of n_1, n_2, &c. We are however able to shew *synthetically* that a certain form given to the numbers n_1, n_2, &c. satisfies the conditions. Let $k_1 = a_1$, $k_2 = a_2$, &c.; $l_1 = b_1$, $l_2 = b_2$, &c.; $m_1 = b_1$, $m_2 = b_2$, &c.; so that the final equations of Article 88 take the form

$$x \,.\, \Sigma(a^2) + y \,.\, \Sigma(ab) + z \,.\, \Sigma(ac) = \Sigma(af) \ldots\ldots [4],$$

$$x \,.\, \Sigma(ab) + y \,.\, \Sigma(b^2) + z \,.\, \Sigma(bc) = \Sigma(bf) \ldots\ldots [5],$$

$$x \,.\, \Sigma(ac) + y \,.\, \Sigma(bc) + z \,.\, \Sigma(c^2) = \Sigma(cf) \ldots\ldots [6].$$

Then the values of x, y, z, which are deduced from these equations, possess the properties required.

92. For, suppose that we obtain the value of x by multiplying the first of these by p, the second by q, the third by r, and taking their sum, the coefficients of y and z being made to vanish. Then

$$x \times \{p \,.\, \Sigma\,(a^2) + q \,.\, \Sigma\,(ab) + r \,.\, \Sigma\,(ac)\}$$
$$= p \,.\, \Sigma\,(af) + q \,.\, \Sigma\,(bf) + r \,.\, \Sigma\,(cf)\,;$$

$$p \,.\, \Sigma\,(ab) + q \,.\, \Sigma\,(b^2) + r \,.\, \Sigma\,(bc) = 0\,;$$

$$p \,.\, \Sigma\,(ac) + q \,.\, \Sigma\,(bc) + r \,.\, \Sigma\,(c^2) = 0\,;$$

which are the same as

$$x \times \Sigma\,\{a\,(pa + qb + rc)\} = \Sigma\,\{f\,(pa + qb + rc)\} \ldots\ldots [7],$$
$$\Sigma\,\{b\,(pa + qb + rc)\} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [8],$$
$$\Sigma\,\{c\,(pa + qb + rc)\} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [9].$$

Comparing these equations with those of Article 89, n is now replaced by $pa + qb + rc$. Therefore

$$\Sigma\,(n^2) = \Sigma\,\{(pa + qb + rc)\,(pa + qb + rc)\}$$
$$= p \,.\, \Sigma\,(an) + q \,.\, \Sigma\{b\,(pa + qb + rc)\} + r \,.\, \Sigma\{c\,(pa + qb + rc)\}.$$

The last two quantities vanish, by virtue of equations [8] and [9]; and therefore $\Sigma\,(n^2) = p \,.\, \Sigma\,(an)$. Substituting this in the first denominator of equation [1], the equation becomes

$$(n_1 - pa_1)\,\delta n_1 + (n_2 - pa_2)\,\delta n_2 + \&c. = 0,$$

or $$(qb_1 + rc_1)\,\delta n_1 + (qb_2 + rc_2)\,\delta n_2 + \&c. = 0\,;$$

or $$\left.\begin{array}{l} \;\;\, q\,(b_1\delta n_1 + b_2\delta n_2 + \&c.) \\ + r\,(c_1\delta n_1 + c_2\delta n_2 + \&c.) \end{array}\right\} = 0 \ldots\ldots\ldots\ldots [10];$$

which is, under the new assumptions, the equivalent of equation [1], and on the truth of which will depend the validity of the new assumptions. Now the equation [10] is true; for its left hand consists of two parts, of which one $=0$ by equation [2], and the other $=0$ by equation [3]. Consequently, the equations [1], [2], [3], are simultaneously satisfied: and therefore the assumption of Article 91 gives the values of x, whose probable error is minimum.

93. If we investigate, by a similar method, the assumption which will give for y the value whose probable error is minimum, we have only to remark that the equations [4], [5], [6], are symmetrical with respect to x, y, and z, and therefore when treated for y in the same manner as for x, they will exhibit the same result for y as for x; that is, the probable error of y, as determined from their solution, is minimum. In the same manner, the probable error of z, as determined from the solution of the same equations, treated in the same manner, is minimum.

The problem, therefore, of determining values of x, y, z, to satisfy, with the smallest probable error of x, y, and z, the numerous equations

$$a_1 x + b_1 y + c_1 z = f_1,$$

$$a_2 x + b_2 y + c_2 z = f_2,$$

&c.

is completely solved by solution of the equations

$$x \,.\, \Sigma\,(a^2) + y \,.\, \Sigma\,(ab) + z \,.\, \Sigma\,(ac) = \Sigma\,(af),$$
$$x \,.\, \Sigma\,(ab) + y \,.\, \Sigma\,(b^2) + z \,.\, \Sigma\,(bc) = \Sigma\,(bf),$$
$$x \,.\, \Sigma\,(ac) + y \,.\, \Sigma\,(bc) + z \,.\, \Sigma\,(c^2) = \Sigma\,(cf).$$

94. Suppose that, instead of proposing to ourselves the condition that the probable errors of the deduced values of x, y, z, shall be minimum, we had proposed this condition; that the sum of the squares of the errors remaining after correction for the deduced values of x, y, and z, or

$$\Sigma \,.\, (ax + by + cz - f)^2,$$

shall be minimum. On differentiating each equation with respect to x, and taking their sum, we should have obtained

$$\Sigma\,\{a \,.\, (ax + by + cz - f)\} = 0\,;$$

and similarly for y and z

$$\Sigma\,\{b \,.\, (ax + by + cz - f)\} = 0,$$
$$\Sigma\,\{c \,.\, (ax + by + cz - f)\} = 0\,;$$

the very same equations as those found above. In consequence of this property of the equations, of giving such values of x, y, and z, that the sum of squares of errors remaining after their application shall be minimum, the method is very frequently called "the method of minimum squares." This term is very unfortunate; it has frequently led investigators to suppose that the subject of

the minimum is the sum of squares of discordances as first presented; whereas it ought to be the sum of squares of discordances, when so multiplied as to have the same probable error.

95. It is easy to see that the same principles apply, the same remarks hold, and the same result is obtained, when the number of unknown elements, instead of being restricted to three, is any whatever. The rule is universal; multiply every equation by such a factor that the probable error of the right-hand term will be the same for all; multiply every altered equation by its coefficient of one unknown quantity, and take the sum for a new equation; the same for the second unknown quantity; and so on for every unknown quantity; and thus a number of equations will be found equal to the number of unknown quantities.

96. In order to exhibit the probable error of x thus determined, we may proceed by a purely algebraical process. It will however soon be found that it leads to results of intolerable complexity. We would recommend the reader to introduce numbers as soon as possible for every symbol except f (that quantity from whose error all errors spring). In the following explanation, however, of the succession of steps, the reader will easily see to what extent he can advantageously retain the symbols.

It is first necessary to find the factors of the equations [4], [5], [6], of Article 91, or the last equations of Article

93, which will eliminate y and z. They are easily found to be,

For equation [4], $\Sigma . b^2 \times \Sigma . c^2 - (\Sigma . bc)^2$.

For equation [5], $\Sigma . ac \times \Sigma . bc - \Sigma . ab \times \Sigma . c^2$.

For equation [6], $\Sigma . ab \times \Sigma . bc - \Sigma . ac \times \Sigma . b^2$.

There is no difficulty in finding the factors when the number of unknown quantities exceeds three; but the trouble is so great that it will always be best to use numbers.

Applying these, we obtain

$$x = P . \Sigma (af) + Q . \Sigma (bf) + R . \Sigma (cf),$$

where P, Q, R, are numbers, but af, bf, cf, are for the present retained in the symbolical form.

Now if we examine the form in which the individual quantities f_1, f_2, &c. enter into this expression, and if we collect together all the multiples of each individual quantity, we shall find

$$x = (Pa_1 + Qb_1 + Rc_1) f_1 + (Pa_2 + Qb_2 + Rc_2) f_2 + \&c.$$

We have here a number of independent fallible quantities, to which the formula of Article 52 will properly apply. Remarking that the probable error of each of the quantities f_1, f_2, &c. is supposed to $= e$, we obtain

$$\left(\frac{\text{p. e. of } x}{e}\right)^2 = (Pa_1 + Qb_1 + Rc_1)^2 + (Pa_2 + Qb_2 + Rc_2)^2 + \&c.$$

$$= \Sigma . (Pa + Qb + Rc)^2;$$

which may be exhibited in symbols of great complexity, but which it will be very far easier to evaluate in numbers by an entirely numerical process.

The operation for finding the probable errors of y and z would be exactly similar.

97. The relaxation of the rules for determining the most advantageous values of the factors of the equations, which in reference to the treatment of simple measures is explained in Article 73, is admissible also in the treatment of equations applying to several unknown quantities, and for the same theoretical reason. By taking advantage of this relaxation, the labour may sometimes be materially diminished. In actual applications, the numbers a_1, a_2, &c. b_1, &c., usually consist of troublesome decimals. In practice, all desirable accuracy will be secured for the result, by striking off, in the factors only, all the latter decimals, leaving only one or two significant figures. The use of different factors will produce different results, but not necessarily more inaccurate results ; we have no means of certainly knowing which are the best; we only know that, if we repeat the process in an infinity of instances, the factors corresponding accurately to minimum will furnish us with results whose errors are, on the whole, a little smaller than those originating from other factors.

§ 14. *Instances of the formation of equations applying to several unknown quantities.*

98. It will perhaps be instructive to shew how equations, such as those treated above, arise. For this purpose, we will take two instances; one of very simple and one of very complicated character.

99. Instance 1. It is required to determine the most probable values of the personal equations between a number of transit observers A, B, C, D, &c.; where the observers have been brought into comparison in many combinations, or perhaps in every possible combination; but never more than two at a time.

Use the symbol (ab) to denote the number of comparisons between A and B, and $A-B$ for the symbolical value of the personal equation between A and B, (AB) for its numerical value deduced from the mean of comparisons. And suppose that the probable error of each single comparison is e. Then the probable error of (AB) is $\frac{e}{\sqrt{(ab)}}$. Therefore when we have formed the equation

$$A-B=(AB),$$

in which the last term is liable to the probable error $\frac{e}{\sqrt{(ab)}}$, we must, in conformity with the recommendation in Article 87, multiply the equation by $\sqrt{(ab)}$, and then

its probable error will be e. Thus we find, for the different comparisons, the following equations, all liable to the same probable error e:

$$\sqrt{(ab)} \, . \, A - \sqrt{(ab)} \, . \, B = \sqrt{(ab)} \, . \, (AB),$$

$$\sqrt{(ac)} \, . \, A - \sqrt{(ac)} \, . \, C = \sqrt{(ac)} \, . \, (AC),$$

&c.

$$\sqrt{(bc)} \, . \, B - \sqrt{(bc)} \, . \, C = \sqrt{(bc)} \, . \, (BC),$$

&c.,

and these equations are exactly such as those in Article 88, though in an imperfect form. The determining equations must therefore be formed by the rule of Article 93. Thus we find;—

The first equation is to be formed by the sum of the following,

$$(ab) \, . \, A - (ab) \, . \, B = + (ab) \, . \, (AB),$$

$$(ac) \, . \, A - (ac) \, . \, C = + (ac) \, . \, (AC),$$

&c.

The second equation is to be formed by the sum of the following,

$$(ab) \, . \, B - (ab) \, . \, A = - (ab) \, . \, (AB),$$

$$(bc) \, . \, B - (bc) \, . \, C = + (bc) \, . \, (BC),$$

&c.

Thus we obtain the simple rule:—

Form each equation for comparison of two observers by taking the mean of all their comparisons.

Multiply each such equation by its number of comparisons. This is, in fact, the same as if the sum of all the individual comparisons of those two observers had been taken.

In the various multiplied equations which contain A; make the coefficient of A in every equation positive (by changing all the signs of the equation where necessary), and then add all together to form a determining equation.

In the various multiplied equations which contain B, including, if necessary, one from the last-mentioned series; make the coefficient of B in every equation positive (by changing signs if necessary), and add all together to form a determining equation.

In the various multiplied equations which contain C, including, if necessary, one from each of the last two series; make the coefficient of C in every equation positive (by changing signs if necessary), and add all together to form a determining equation.

And so through all the observers.

It will be found that one of the determining equations may be produced by a combination of all the other determining equations; and therefore it is necessary to assume a value for one of the quantities, A, or B, or C, &c.

100. Instance 2. In a net (not necessarily a simple chain) of geodetic triangles; one or more sides have been actually measured, or so determined by immediate reference to measured bases that they may be considered as measured; in some of the triangles, three angles have been measured, in others only two; at some stations, all the angles round the circle have been observed, at others not all; at some stations, astronomical azimuths have been observed: it is required to lay down the rules for determination of the positions of the different stations.

101. It is first necessary to determine the value of probable error in each of the observations. And this is not to be done by a simple rule, because the observations are not all alike. For instance, the horizontal angle between two signals is liable to error from (1) error of instrument, (2) error of pointing to one signal, (3) error of pointing to the other signal; and when each probable error is ascertained, the probable error of horizontal angle between signals is easily formed. But for the azimuth of a given signal, the sources of error are, (1) error of instrument, (2) error of pointing to the signal, (3) error of pointing in the direction of the meridian; and the probable error of this last may be very different from the others. The linear measures will require a peculiar estimate of probable error.

All must, however, antecedent to all other treatment, be so found that the probable error of every measure, of whatever kind, can be specified.

102. The next step in this (and in all other complicated cases) will be, to assume that every co-ordinate of station which we are seeking, is approximately known, and numerically expressed. Thus, if the triangulation is so small in scale that its area may be supposed a plane, we may assume, for the two rectangular co-ordinates of every point, numerical values, each subject to a small correction (which corrections it is the object of the whole investigation to ascertain). If the triangulation is so large in scale that the spheroidal form of the earth must be regarded, we may assume, for the astronomical latitude and longitude of every point, numerical values, each subject to a small correction.

103. With these numerical values and symbolical corrections, every fact which has been the subject of measure must be computed; and the computation-result must consist of two parts, one numerical, and the other multiplying the symbolical corrections. Thus; suppose that the area is plane, and that the rectangular co-ordinates are

For 1st station, $a_1 + \delta a_1$, $b_1 + \delta b_1$,

For 2d station, $a_2 + \delta a_2$, $b_2 + \delta b_2$,

For 3d station, $a_3 + \delta a_3$, $b_3 + \delta b_3$,

&c.

(where a_1, a_2, &c., b_1, b_2, &c., are numerical, and δa_1, δa_2, &c., δb_1, δb_2, &c. are symbols only). Suppose that the direction of a is parallel to the meridian. Then the azimuth of the second station as viewed from the first is

$$C_2 + \frac{a_2 - a_1}{(a_2 - a_1)^2 + (b_2 - b_1)^2}(\delta b_2 - \delta b_1)$$

$$- \frac{b_2 - b_1}{(a_2 - a_1)^2 + (b_2 - b_1)^2}(\delta a_2 - \delta a_1),$$

where $\tan C_2 = \frac{b_2 - b_1}{a_2 - a_1}$, and where all quantities are numerical, except δa_1, δb_1, δa_2, δb_2. For convenience we will write this,

True azimuth of 2d station as seen from 1st station

$$= C_2 + A_{1,2} . \delta a_2 - A_{1,2} . \delta a_1 + B_{1,2} . \delta b_2 - B_{1,2} . \delta b_1.$$

And in like manner,

True azimuth of 3d station as seen from 1st station

$$= C_3 + A_{1,3} . \delta a_3 - A_{1,3} . \delta a_1 + B_{1,3} . \delta b_3 - B_{1,3} . \delta b_1 .$$

Now if the azimuth of the 2d station had been observed at the 1st, and found $= \gamma_2$ (subject to error of observation), then the comparison of the first formula with this would give the equation

$$A_{1,2} . \delta a_2 - A_{1,2} . \delta a_1 + B_{1,2} . \delta b_2 - B_{1,2} . \delta b_1 = \gamma_2 - C_2.$$

If, however, no azimuth from the meridian had been observed, but only the horizontal angle $\theta_{2,3}$ between the 2d and 3d stations, which is the same as $\gamma_3 - \gamma_2$, and is subject to errors of observation; then we should have

$$A_{1,3}.\delta a_3 - A_{1,2}.\delta a_2 + (A_{1,2} - A_{1,3}).\delta a_1 + B_{1,3}.\delta b_3 - B_{1,2}.\delta b_2$$
$$+ (B_{1,2} - B_{1,3}).\delta b_1$$
$$= \theta_{2,3} - C_3 + C_2.$$

The distance of the 2d station from the first

$$= \sqrt{\{(a_2 - a_1)^2 + (b_2 - b_1)^2\}}$$

$$+ \frac{a_2 - a_1}{\sqrt{\{(a_2 - a_1)^2 + (b_2 - b_1)^2\}}} (\delta a_2 - \delta a_1)$$

$$+ \frac{b_2 - b_1}{\sqrt{\{(a_2 - a_1)^2 + (b_2 - b_1)^2\}}} (\delta b_2 - \delta b_1),$$

which may be written

$$L_2 + M_{1,2} (\delta a_2 - \delta a_1) + N_{1,2} (\delta b_2 - \delta b_1).$$

Now if the distance from the first station to the second had been measured, and found $= \lambda_2$, subject to error of observation; then the comparison with this formula would give

$$M_{1,2} . \delta a_2 - M_{1,2} . \delta a_1 + N_{1,2} . \delta b_2 - N_{1,2} . \delta b_1 = \lambda_2 - L_2.$$

104. Each of these equations contains, on the right hand, a fallible quantity; the first contains γ_2, the second contains $\theta_{2,3}$, the third contains λ_2. The probable error of each of these, as we have said in Article 101, must be supposed known; and then, the equation must be divided by a divisor proportional to that probable error. This operation being effected, we shall have a series of equations whose probable errors are all equal; and the rule of Article 93 can be applied; and we shall have a series of determining equations equal in number to the number of unknown quantities; that is, to double the number of stations.

105. The reader who will well examine such an instance as this will be struck with the perfect generality and great beauty of the theory. He will see that, whatever has been observed, and in whatever shape, as a fact of observation, will enter with its proper weight into the formation of the final determining equations. He may exercise his fancy in introducing different circumstances. Suppose, for instance, that a referring-signal has been used. The correction of assumed azimuth of that referring-signal will be a new unknown quantity. Sometimes it is combined only with observations of signals, in which case it produces one form of equation; sometimes it is combined also with meridian-observations, in which case it produces another form of equation. In some batches of observations, it may be necessary to use the theory of "entangled measures" (see Articles 74 to 86) before the probable errors can be properly found. Whatever measure is made, a proper corresponding equation can be formed, and the proper cautions accompanying it will soon present themselves.

106. It may occur to the reader as a difficulty, that the quantities concerned are not homogeneous; the measure of θ, for instance, being in angle, and that of λ in units of length. This, however, is only apparent. Any measure is expressed by means of certain units, and its probable error is expressed by the same units; and, when we divide each equation by a divisor proportional to its probable error, we do in fact produce an equation of abs-

tract numbers, whose probable error also is an abstract number. The whole are then strictly comparable, in whatever kinds of measure they have originated.

107. The solution of so numerous a series of equations is of course troublesome. It is, however, no more troublesome than the nature of the subject strictly requires. And it is to be considered that it gives to every observation of every kind exactly the weight that is due to it: that it leads to one distinct system of results, without leaving any opening for uncertainty; and that that system of results is the most probable.

§ 15. *Treatment of Observations in which it is required that the Errors of Observations rigorously satisfy some assigned conditions.*

108. In the equations considered in Article 87, each of which gave the effects of combining, with different factors, the unknown corrections of certain elements, in order that observations might thereby be better represented,—it is to be remarked that there was no expectation that the result of combination of these corrections of elements would *exactly* represent the observations, or that any *exact* relation could be assigned as existing among the corrections which were to be found; or between "the result of applying those corrections," and "observations".

109. But there are instances in which the nature of the problem requires that, among the corrections which

are to be found, a prescribed equation shall be rigorously maintained. The nature and treatment of these will best be understood from examples.

110. Instance (1). In a geodetic triangle, observations of the three angles are made. On comparing their sum with the quantity $180^0 +$ spherical excess, to which that sum ought to be equal, it is found that the sum requires the correction A. How ought that correction to be divided among the three angles, their probable errors of observation being known?

111. Let E_1, E_2, E_3 be the three corrections required, the probable errors of observation being e_1, e_2, e_3. Then the first consideration is,

E_1 is to be as small as possible,
E_2 ....,
E_3 ....

The quantity E_1 here corresponds to the quantity

$$ax + by + cz - f$$

in Article 87; the latter being the residual difference between an observed quantity and a quantity computed from corrected elements, which difference is to be made "as small as possible." The equations are therefore to be exhibited in the same form.

Hence we may state the equations thus:

$E_1 = 0$, with probable error e_1,
$E_2 = 0$, ... e_2,
$E_3 = 0$, ... e_3.

If we stopped at this form, we could not obtain a valid solution: the number of equations being the same as the number of unknown quantities, in which case no solution depending on probabilities can be obtained.

112. Now we introduce the condition

$$E_1 + E_2 + E_3 = A,$$

and use it to eliminate one of the quantities, as E_3. Then the equations become,

$$\begin{aligned} E_1 &= 0, \text{ with probable error } e_1, \\ E_2 &= 0, \text{ .... } e_2, \\ A - E_1 - E_2 &= 0, \text{ .... } e_3. \end{aligned}$$

Here we have three equations to determine two quantities, and the process of Article 93 may be followed.

113. Dividing by the probable errors, we have these equations, in each of which the probable error $= 1$;

$$\frac{E_1}{e_1} = 0,$$

$$\frac{E_2}{e_2} = 0,$$

$$\frac{A}{e_3} - \frac{E_1}{e_3} - \frac{E_2}{e_3} = 0,$$

and therefore, by the process of Article 93, forming a final equation principally for E_1, by multiplying each

equation here by its coefficient of E_1 and adding the products,

$$\frac{E_1}{(e_1)^2} - \frac{A}{(e_3)^2} + \frac{E_1}{(e_3)^2} + \frac{E_2}{(e_3)^2} = 0.$$

Similarly, forming a final equation principally for E_2, by multiplying each equation by its coefficient of E_2, and adding the products,

$$\frac{E_2}{(e_2)^2} - \frac{A}{(e_3)^2} + \frac{E_1}{(e_3)^2} + \frac{E_2}{(e_3)^2} = 0.$$

Comparing these two equations, $\frac{E_1}{(e_1)^2} = \frac{E_2}{(e_2)^2}$. We might at once infer from this that $\frac{E_3}{(e_3)^2}$ has the same value, but it may be more satisfactory to solve the equations completely. Eliminating E_2 from the first equation, by the relation just found,

$$E_1 \left\{ \frac{1}{(e_1)^2} + \frac{1}{(e_3)^2} + \frac{(e_2)^2}{(e_1 \cdot e_3)^2} \right\} - \frac{A}{(e_3)^2} = 0;$$

or $$E_1 \{(e_3)^2 + (e_1)^2 + (e_2)^2\} - A\,(e_1)^2 = 0.$$

Therefore $$E_1 = A \frac{(e_1)^2}{(e_1)^2 + (e_2)^2 + (e_3)^2};$$

whence, by the relation found,

$$E_2 = A \frac{(e_2)^2}{(e_1)^2 + (e_2)^2 + (e_3)^2};$$

and by subtracting their sum from A,

$$E_3 = A \frac{(e_3)^2}{(e_1)^2 + (e_2)^2 + (e_3)^2}.$$

Hence, the corrections to be assigned to the different angles ought to be proportional to the squares of their respective probable errors.

114. Instance (2). From a theodolite station, n signals can be seen; the angles, between each signal and the next in azimuth, are independently observed; their sum, which ought to be 360°, is $360^\circ - B$: how ought the correction B to be divided?

115. The equations in this instance will be

$E_1 = 0$, with probable error e_1,

$E_2 = 0$, ... e_2,

....

....

$E_n = 0$, ... e_n.

Then, by the equation

$$E_1 + E_2 + \&c. + E_n = B,$$

the last of the equations is changed into

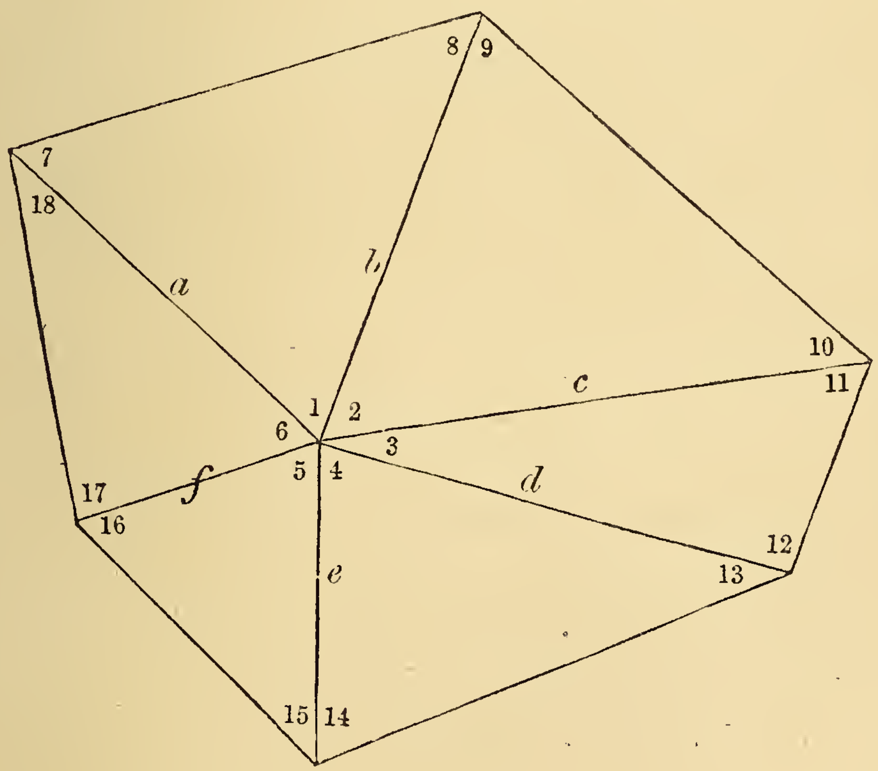
$B - E_1 - E_2 - \&c. - E_{n-1} = 0$, with probable error e_n;

and the equations are to be treated in the same manner as in Instance (1); and a similar result is obtained; namely,

that the corrections to be assigned to the different angles must be proportional to the squares of their respective probable errors.

The next instance will be more complicated.

116. Instance (3). In the survey of a chain of triangles, a hexagonal combination of the following kind occurs, in which every angle is observed independently; all are liable to error; to find the correction which ought to be made to each.



Let the angles be denoted by the simple numbers; let their corrections sought be [1], [2], &c., and their probable errors of observation (1), (2), &c.

Then we have the equations,

$$[1] = 0, \text{ with probable error } (1),$$
$$[2] = 0, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (2);$$
$$\ldots\ldots \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$[18] = 0, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (18).$$

And now we have to consider how many of these unknown quantities can be eliminated by virtue of the geometrical relations.

117. Adding the angles at the central station, and comparing the sum with 360°,

$$[1] + [2] + [3] + [4] + [5] + [6] = \text{a known quantity } A.$$

Then in the six triangles,

$$[1] + [7] + [8] = \text{a known quantity } B,$$
$$[2] + [9] + [10] \ \ldots\ldots\ldots\ldots\ldots\ldots \ C,$$
$$[3] + [11] + [12] \ \ldots\ldots\ldots\ldots\ldots\ldots \ D,$$
$$[4] + [13] + [14] \ \ldots\ldots\ldots\ldots\ldots\ldots \ E,$$
$$[5] + [15] + [16] \ \ldots\ldots\ldots\ldots\ldots\ldots \ F,$$
$$[6] + [17] + [18] \ \ldots\ldots\ldots\ldots\ldots\ldots \ G.$$

When corrections satisfying these equations are applied, we shall have a set of six triangles, with angles consistent in each triangle; and which so adhere together that they fill up 360° at the central station; nevertheless

it might happen that, in calculating b from a, c from b, d from c, e from d, f from e, and a' from f, we should find a value a' differing from a. But it is necessary that a' be found rigorously equal to a. Tracing the calculations through the several triangles, it is found that this equation gives (with corrected angles),

$$\frac{\sin 7}{\sin 8} \times \frac{\sin 9}{\sin 10} \times \frac{\sin 11}{\sin 12} \times \frac{\sin 13}{\sin 14} \times \frac{\sin 15}{\sin 16} \times \frac{\sin 17}{\sin 18} = 1,$$

and, taking the logarithms, with the addition of symbols for the corrections,

$$\begin{aligned}
&\log.\sin 7 - \log.\sin 8 + \log.\sin 9 - \log.\sin 10 + \log.\sin 11 \\
&- \log.\sin 12 + \log.\sin 13 - \log.\sin 14 + \log.\sin 15 - \log.\sin 16 \\
&\qquad + \log.\sin 17 - \log.\sin 18 \\
&+ \cot 7 \times [7] - \cot 8 \times [8] + \cot 9 \times [9] - \&c. \ldots\ldots \\
&\ldots\ldots + \cot 17 \times [17] - \cot 18 \times [18] \\
&= 0.
\end{aligned}$$

We shall use the symbol L to denote the first part of this expression, which is a known quantity.

Thus we have eight equations to be rigorously satisfied. By means of these, we are to eliminate eight of the quantities [1], [2], &c., and there will remain ten quantities to be determined by eighteen equations.

118. Suppose for instance we decide to eliminate the corrections [1], [2], &c., as far as [8]. We have

$$[2] = C - [9] - [10],$$
$$[3] = D - [11] - [12],$$

$$[4] = E - [13] - [14],$$
$$[5] = F - [15] - [16],$$
$$[6] = G - [17] - [18].$$

Substituting these in the first equation,

$$[1] = A - C - D - E - F - G$$
$$+ [9] + [10] + [11] + [12] + [13] + [14] + [15] + [16] + [17] + [18].$$

Then

$$\begin{aligned}[7] &= B - [1] - [8] \\ &= -A + B + C + D + E + F + G \\ &\quad - [8] - [9] - [10] - [11] - [12] - [13] - [14] - [15] \\ &\quad - [16] - [17] - [18].\end{aligned}$$

Substituting this in the last equation of Article 117,

$$\begin{aligned}0 = {} & \\ L &+ \cot 7 \times (-A + B + C + D + E + F + G) \\ &- \cot 7 \times \{[8] + [9] + [10] + [11] + [12] + [13] + [14] + [15] \\ &\qquad + [16] + [17] + [18]\} \\ &- \cot 8 \times [8] + \cot 9 \times [9] - \cot 10 \times [10] + \cot 11 \times [11] \\ &- \cot 12 \times [12] + \cot 13 \times [13] - \cot 14 \times [14] \\ &+ \cot 15 \times [15] - \cot 16 \times [16] + \cot 17 \times [17] \\ &- \cot 18 \times [18].\end{aligned}$$

From this equation, [8] is found in terms of [9], [10], &c. as far as [18]. And substituting it in the preceding expression, [7] is found in terms of [9], [10], &c. as far as [18]. Thus all the corrections [1], [2], ... [8], are expressed in terms of [9], [10] ... [18].

119. The primary equations of probabilities are,

$$[1] = 0 \text{ with probable error } (1),$$
$$[2] = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$[18] = 0 \ldots\ldots\ldots\ldots\ldots\ldots (18).$$

Of these, the first eight will now be changed into the following:

$$[9] + [10] + \ldots + [18] = -A + C + D + E + F + G,$$
with probable error (1),

$$[9] + [10] = C, \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$[11] + [12] = D, \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$
$$[13] + [14] = E, \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$
$$[15] + [16] = F, \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$
$$[17] + [18] = G, \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$\left\{\begin{array}{c}\text{series of multiples}\\ \text{of } [9], [10] \ldots [18],\\ \text{expressing } [7]\end{array}\right\} = \left\{\begin{array}{c}\text{series of}\\ \text{known}\\ \text{quantities}\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$\left\{\begin{array}{c}\text{series of multiples}\\ \text{of } [9], [10] \ldots [18],\\ \text{expressing } [8]\end{array}\right\} = \left\{\begin{array}{c}\text{series of}\\ \text{known}\\ \text{quantities}\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

The remaining equations will retain their simple form,

$$[9] = 0, \text{ with probable error } (9),$$
$$[10] = 0, \ldots\ldots\ldots\ldots\ldots\ldots (10),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$[18] = 0, \ldots\ldots\ldots\ldots\ldots\ldots (18).$$

120. Each of these eighteen equations is then to be divided by its probable error, and we thus obtain the following equations, in each of which the probable error $=1$;

$$\frac{[9]}{(1)}+\frac{[10]}{(1)}\cdots\cdots+\frac{[18]}{(1)}=\frac{-A+C+D+E+F+G}{(1)},$$

with probable error 1.

$$\frac{[9]}{(2)}+\frac{[10]}{(2)}=\frac{C}{(2)}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1,$$

and so through all the equations.

The equations so divided, having the same probable error, are in a fit state for application of the method of Article 93. The first of the final equations (principally for [9]) will be formed by multiplying each equation by the coefficient of [9] in that equation, and adding the products; the second of the final equations (principally for [10]) will be formed by multiplying each equation by the coefficient of [10] in that equation, and adding the products; and so on to [18]. From the equations thus formed, the values of [9], [10]...[18], are found; and by substituting these in the formulæ of Article 118, the values of [1], [2]...[8] are found.

121. It is particularly to be observed that, although in the changed equations of probabilities we eliminate such quantities as [1], [2], &c., we do not eliminate their corresponding probable errors (1), (2), &c., each of which must be left in its place. This retention of the probable error will be remarked in Instances (1) and (2).

122. The complete solution is so troublesome that it would scarcely ever be used in practice. Probably some process like the following would be employed, with sufficient accuracy:

Divide the error A by the process of Instance (2), and use the corrected angles in the process that follows.

Divide the errors $B, C, \ldots G$, by the process of Instance (1), and use the corrected angles in the process that follows.

Apply the last equation of Article 117, by a process nearly similar to that for A.

Repeat the process for dividing A' (the discordance at the center produced by the angles as last corrected).

Repeat the process for dividing $B', C' \ldots G'$. And continue this operation as often as may be necessary.

PART IV.

ON MIXED ERRORS OF DIFFERENT CLASSES, AND CONSTANT ERRORS.

§ 16. *Consideration of the circumstances under which the existence of Mixed Errors of Different Classes may be recognized, and investigation of their separate values.*

123. WHEN successive series of observations are made, day after day, of the same measurable quantity, which is either invariable (as the angular distance between two components of a double star) or admits of being reduced by calculation to an invariable quantity (as the apparent angular diameter of a planet); and when every known instrumental correction has been applied (as for zero, for effect of temperature upon the scale, &c.); still it will sometimes be found that the result obtained on one day differs from the result obtained on another day by a larger quantity than could have been anticipated. The idea then presents itself, that possibly there has been on some one day, or on every day, some cause, special to the day, which has produced a *Constant Error* in the measures of that day. It is our business now to consider the evidence for, and the treatment of, such constant error.

124. The existence of a daily constant error, that is, of an additional error which follows a different law from the ordinary error, ought not to be lightly assumed. When observations are made on only two or three days, and the number of observations on each day is not extremely great, the mere fact, of accordance on each day and discordance from day to day, is not sufficient to prove a constant error. The existence of an accordance analogous to a "run of luck" in ordinary chances is sufficiently probable. If this be accepted, as applying to each day, the whole of the observations on the different days must be aggregated as one series, subject to the usual law of error. More extensive experience, however, may give greater confidence to the assumption of constant errors; and then the treatment of which we proceed to speak will properly apply.

125. First, it ought, in general, to be established that there is possibility of error, constant on one day but varying from day to day. Suppose, for instance, that the distance of two near stars is observed with some double-image instrument by the method of three equal distances, alternately right and left. It does not appear that any atmospherical or personal circumstance can produce a constant error; and, unless we are driven to it by considerations like those to be mentioned in Article 129, we must not entertain it. But suppose, on the other hand, that we have measured the apparent diameter of Jupiter. It is evident that both atmospheric and personal circumstances may

sensibly alter the measure; and here we may admit the possibility of the error.

126. Now let us take the observations of each day separately, and, by the rules of Articles 60 and 61, investigate from each separate day the probable error of a single measure. We may expect to find different values (the mere paucity of observations will sufficiently explain the difference); but as the individual observations on the different days either are equally good, or (as well as we can judge) have such a difference of merit that we can approximately assign the proportion of their probable errors, we can define the value of probable error for observations of standard quality as determined from the observations of each day; we must then combine these, with greater weight for the deductions from the more numerous observations, and we shall have a final value of probable error of each individual observation, not containing the effects of Constant Error. From this we can, by the rule of Article 55, infer the "Probable Error of Each Day's Result;" still not containing the effects of Constant Error. The "Result of Each Day," also not containing any correction for Constant Error, is given by the mean of determinations for each day.

127. We must now attach to the numerical value of "Result of Each Day" a symbol for "Actual Error of Result of Each Day;" and take the mean of all these compound quantities, numerical and symbolical, for all the days; (the combination-weights being proportional to the

number of observations on each day, unless any modifying circumstance require a different proportion). This mean may be regarded as "Final Result." The "Final Result" is to be subtracted from the "Result of Each Day;" the remainder is the "Discordance of Each Day's Result." For each day it consists of two parts; a number, and a series of multiples of all the symbols for "Actual Error of Result of Each Day."

128. Now treat the Discordance (consisting of the number accompanied with multiples of symbols) as being itself an Error, and investigate the "Mean Discordance" by the rule of Article 26 or 59; a value of "Mean Discordance" will thus be obtained, consisting of a number accompanied with a series of multiples of symbols of "Actual Error." Consider each day's "Actual Error" as an independent fallible quantity whose Probable Error is that obtained in Article 126, and form the "Probable Error of Mean Discordance" by the rule of Article 52. Thus we have, for Mean Discordance, a formula consisting of two parts, namely,

(1) A numerical value.

(2) A number expressing the probable error in the determination of that numerical value.

129. And now it will rest entirely in the judgment of the computer to determine whether the simple numerical value (1) just found, is to be adopted for Mean Discordance or not. It is quite clear that, if (2) exceeds (1), there is no

sufficient justification for the assumption of a Discordance, that is, of a Constant Error. If (2) is much less than (1), it appears equally clear that a Constant Error must be assumed to exist, and (1) or any value near it may be adapted for Mean Discordance. The Probable Discordance, or Probable Constant Error, will be found by multiplying this by 0·8453, as in Article 31.

130. The reader must not be startled at our referring these decisions to his judgment, without material assistance from the Calculus. The Calculus is, after all, a mere tool by which the decisions of the mind are worked out with accuracy, but which must be directed by the mind. In deciding on the admissibility of Constant Error, after giving full weight to the considerations of Article 129, it will still be impossible, and would be wrong, to exclude the considerations of Article 125, and these cannot be brought under algebraical or numerical rule.

131. These investigations suppose that the "Discordance of Each Day's Result" cannot, so far as we know antecedently, be referred to any distinct assignable cause. But if there should appear to be any such cause, as, for instance, if we conceive that the observations of one person always give a greater measure than the observations of another person, it will be easy to apply an investigation, analogous to that just given. The observations of each person should be separated from those of other persons and collected together; from the collected group of each person's observations, a Mean Result and Probable Error of

Mean Result for each person must be found; and then the reader must judge whether, in view of the amount of Probable Errors, a Personal Difference of Results is admissible or required. The investigation is simpler than the preceding, in this respect, that it arrives at a Simple Personal Difference of Results, and not at a Mean Discordance. And the result is simpler than the last, because it is a Constant Correction to the results of one person, instead of an uncertain correction liable to the laws of chance.

§ 17. *Treatment of observations when the values of Probable Constant Error for different groups, and probable error of observation of individual measures within each group, are assumed as known.*

132. When numerous and extensive series of observations have been made, as in Articles 126, &c., sufficient to determine the Probable Value of the so-called Constant Error (which is in fact an Error varying from group to group) and the ordinary probable error of an individual observation in each group; suppose that there are made occasional observations, in limited groups, for which it is desirable to define the rules of combination. We are not justified, for each of these limited groups, in assuming a value for the Constant Error, or Variable Error of the Second Class, applicable to that group; we must treat it as an uncertain quantity, and ascertain the combination-weights, and the probable error and theoretical weight of final result, under the effects of the errors of the two classes,

by an operation analogous to those which are applied when the errors are only of one class.

133. In the first group of observations, let the actual value of the error of second class be ${}_1C$; in the second group ${}_2C$; in the third group ${}_3C$, &c.; the probable value of each being c. And in the first group, let the actual values of the errors of first class (or ordinary errors) for the successive observations be ${}_1E_1$, ${}_1E_2$, ${}_1E_3$, &c.; for those in the second group ${}_2E_1$, ${}_2E_2$, &c.; the probable value of each being e. And let the number of observations in the successive groups be ${}_1n$, ${}_2n$, &c. Let the combination factors be ${}_1z_1$, ${}_1z_2$, ${}_1z_3$, &c.; ${}_2z_1$, ${}_2z_2$, ${}_2z_3$, &c.; ${}_3z_1$, ${}_3z_2$, ${}_3z_3$, &c.; and so for successive groups.

Then the actual errors of the separate measures will be

$$
\begin{gathered}
{}_1C + {}_1E_1, \\
{}_1C + {}_1E_2, \\
{}_1C + {}_1E_3, \\
\&c. \\
{}_2C + {}_2E_1, \\
{}_2C + {}_2E_2, \\
{}_2C + {}_2E_3, \\
\&c. \\
{}_3C + {}_3E_1, \\
\&c.
\end{gathered}
$$

And the actual error of the final result, obtained by combining the separate measures with the combination-weights above given, will be the fraction, whose numerator is

$$({}_1z_1 + {}_1z_2 + {}_1z_3 + \&c.)\ {}_1C + ({}_2z_1 + {}_2z_2 + \&c.)\ {}_2C$$
$$+ ({}_3z_1 + {}_3z_2 + \&c.)\ {}_3C + \&c.$$
$$+ ({}_1z_1 \cdot {}_1E_1 + {}_1z_2 \cdot {}_1E_2 + {}_1z_3 \cdot {}_1E_3 + \&c.)$$
$$+ ({}_2z_1 \cdot {}_2E_1 + {}_2z_2 \cdot {}_2E_2 + \&c.)$$
$$+ ({}_3z_1 \cdot {}_3E_1 + {}_3z_2 \cdot {}_3E_2 + \&c.) + \&c.$$

and whose denominator is

$$({}_1z_1 + {}_1z_2 + {}_1z_3 + \&c.) + ({}_2z_1 + {}_2z_2 + \&c.) + ({}_3z_1 + {}_3z_2 + \&c.) + \&c.$$

134. The square of the probable error of the final result, found in exactly the same way as in all preceding cases, will be the fraction whose numerator is

$$({}_1z_1 + {}_1z_2 + \&c.)^2 . c^2 + ({}_2z_1 + {}_2z_2 + \&c.)^2 . c^2 + ({}_3z_1 + {}_3z_2 + \&c.)^2 . c^2 + \&c.$$
$$+ \{({}_1z_1)^2 + ({}_1z_2)^2 + \&c.\}\ e^2 + \{({}_2z_1)^2 + ({}_2z_2)^2 + \&c.\}\ e^2$$
$$+ \{({}_3z_1)^2 + ({}_3z_2)^2 + \&c.\}\ e^2 + \&c.$$

and whose denominator is

$$\{({}_1z_1 + {}_1z_2 + \&c.) + ({}_2z_1 + {}_2z_2 + \&c.) + ({}_3z_1 + {}_3z_2 + \&c.) + \&c.\}^2.$$

This is to be made minimum with respect to the variation of each of the quantities ${}_1z_1$, ${}_1z_2$, &c., ${}_2z_1$, ${}_2z_2$, &c., ${}_3z_1$, ${}_3z_2$, &c. &c. Differentiating with respect to each, making each differential coefficient $= 0$, and treating as in former instances, we find successively, (putting A for an indeterminate constant),

First, $${}_1z_1 = {}_1z_2 = {}_1z_3 = \&c.$$

therefore, for each of these we may use the symbol ${}_1z$.

Second, $${}_1n \,.\, {}_1z \,.\, c^2 + {}_1z \,.\, e^2 = A,$$
$${}_2n \,.\, {}_2z \,.\, c^2 + {}_2z \,.\, e^2 = A,$$
$${}_3n \,.\, {}_3z \,.\, c^2 + {}_3z \,.\, e^2 = A,$$
$$\&c.$$

from which we obtain

$${}_1z = \frac{A}{{}_1n \,.\, c^2 + e^2},$$

which is applicable to every observation in the first group;

$${}_2z = \frac{A}{{}_2n \,.\, c^2 + e^2},$$

which is applicable to every observation in the second group; and so on through all the groups.

135. In the numerator of expression for the square of probable error of result, if for ${}_1z_1$, ${}_1z_2$, &c., we insert ${}_1z$, and so for other groups, it becomes

$${}_1n^2 \,.\, {}_1z^2 \,.\, c^2 + {}_2n^2 \,.\, {}_2z^2 \,.\, c^2 + \&c. + {}_1n \,.\, {}_1z^2 \,.\, e^2 + {}_2n \,.\, {}_2z^2 \,.\, e^2 + \&c.$$
$$= A\,({}_1n \,.\, {}_1z + {}_2n \,.\, {}_2z + \&c.),$$

and the same substitution converts the denominator to

$$({}_1n \,.\, {}_1z + {}_2n \,.\, {}_2z + \&c.)^2;$$

and the square of probable error of result

$$= \frac{A}{{}_1n \,.\, {}_1z + {}_2n \,.\, {}_2z + \&c.};$$

which with the values of ${}_1z$, ${}_2z$, &c. found above, becomes

$$\frac{1}{\dfrac{{}_1n}{{}_1n \, . \, c^2 + e^2} + \dfrac{{}_2n}{{}_2n \, . \, c^2 + e^2} + \&c.}.$$

Or $$\frac{1}{(\text{probable error of result})^2} = \frac{1}{c^2 + \dfrac{e^2}{{}_1n}} + \frac{1}{c^2 + \dfrac{e^2}{{}_2n}} + \&c.$$

136. If, as in Article 131, we conceive that we can fix upon some distinct cause of Constant Error for one group, all the others being assumed free from Constant Error, and can ascertain with confidence the amount of the Constant; that group of results may then be reduced by application of the Constant. For the determination of the probable error of the result of the group so corrected, it must be borne in mind that the determination of the Constant is liable to error. Let A, B, C, D, &c. to n terms, be the actual errors, and a, b, c, d, &c. the probable errors of the means of various groups, A corresponding to that in which we suspect sufficient reason for assuming a Constant Error. The actual error of determination of Constant Error will be

$$A - \frac{B + C + D + \&c.}{n - 1},$$

and the probable error of determination of Constant Error will be

$$\sqrt{\left\{a^2 + \frac{b^2 + c^2 + d^2 + \&c.}{(n-1)^2}\right\}}.$$

But the corrected result of that group will be liable to the actual error

$$A - \left\{A - \frac{B + C + D + \&c.}{n-1}\right\} = \frac{B + C + D + \&c.}{n-1},$$

and its probable error will be

$$\sqrt{\left\{\frac{b^2 + c^2 + d^2 + \&c.}{(n-1)^2}\right\}}.$$

In fact, by referring that result to the mean of other results, and so determining its correction, we entirely deprive that result of any original value in the application of these groups.

But if, on another occasion, there were observations made by the same person or under the same circumstances as the observations A, then the determination of Constant Error and of its probable error just found would be properly applicable.

These conclusions will be varied according to the various assumptions made; the reader will have little difficulty in applying the theory of preceding Articles to any of them.

CONCLUSION.

137. In the practical applications of the Theory of Errors of Observations and of the Combination of Observations which have fallen under our notice, the following are the principal sources of error and inconvenience.

(1) In some instances, measures have been combined by a method of "minimum squares" without reference to the value of probable error of each of the separate observations; and an erroneous result has been deduced. The computer, apparently, has had his attention engrossed by "minimum squares" as the important result to be obtained; whereas, in reality, the satisfying of the equations for minimum squares produces a merely accidental coincidence of results in certain cases (not in all) with those leading to the "minimum probable error of final result," which is the legitimate object of search.

(2) In some instances, entangled observations have been treated as if they were independent, and an erroneous result has been inferred.

(3) In some instances, the labour of application of the theory has been greatly and unnecessarily increased by the use of numerical coefficients proceeding to several places of decimals; when simple factors would have given results possessing all desirable accuracy.

We believe that, by avoiding these errors, and by otherwise conforming to the principles of this Treatise, the Theory of the Combination of Observations may, without great labour, be made a valuable aid in the computation of Physical Measures.

APPENDIX.

PRACTICAL VERIFICATION OF THE THEORETICAL LAW FOR THE FREQUENCY OF ERRORS.

WITH the view of examining the practical accuracy of the formula $A \times \epsilon^{-\frac{x^2}{c^2}} . \delta x$ (see Article 24 and preceding Articles) for the frequency of the occurrence of Errors of Measure &c. between the error-magnitudes x and $x + \delta x$, I collected the results (636 in number) of all the observations of the N.P.D. of Polaris made at the Royal Observatory in the years 1869 to 1873, as reduced in each year to exhibit the mean N.P.D. at the beginning of the year from every observation in the year. In every separate year the difference between each of these mean N.P.D. and the annual mean of all was taken. From the large number of observations in each year, and from the perfect certainty as to the elements of reduction from one year to another, it was evident that there would be no appreciable error in considering these "differences from the mean" in each year as identical with the "differences from the mean" of all which would have been obtained if all the results had been referred to one epoch of time and treated in one group. I therefore extracted from the various groups all the "differences from the mean," and arranged them in order of magnitude, from the largest negative "difference" $-2''\cdot35$ to the largest positive "difference" $+3''\cdot51$. These may be considered as veritable errors of observation. The sum of negative errors was $-215''\cdot88$, and the sum of positive errors $+213''\cdot73$ (the

small discordance between them arising from the loss of figures in the succeeding decimal places). The mean error or $\frac{215''\cdot 88 + 213''\cdot 73}{636}$ was $= 0''\cdot 6755$: from which, by the table in Article 31, the Modulus was found to be $1''\cdot 1973$, and the Probable Error $0''\cdot 5711$.

The errors were then divided into small groups, each group extending over an error-range of $0''\cdot 05$; from $0''\cdot 03$ to $0''\cdot 07$, $0''\cdot 08$ to $0''\cdot 12$, $0''\cdot 13$ to $0''\cdot 17$, and so on, both in the positive and in the negative direction. But, as the frequency of errors for the large values was very small, one group was extended from $-2''\cdot 38$ to $-2''\cdot 13$, one from $-2''\cdot 12$ to $-1''\cdot 98$, one from $-1''\cdot 97$ to $1''\cdot 73$, one from $-1''\cdot 72$ to $-1''\cdot 58$, one from $+1''\cdot 58$ to $+1''\cdot 72$, one from $+1''\cdot 73$ to $+1''\cdot 97$, one from $+1''\cdot 98$ to $+2''\cdot 12$, one from $+2''\cdot 13$ to $+2''\cdot 38$, one from $+2''\cdot 39$ to $+3''\cdot 58$. grouping

The only result extracted from these groups was, the number of observations in each group: and this was considered as representing the frequency through an error-range of $0''\cdot 05$, corresponding in formula to the magnitude of the central error of the group as the independent variable: thus the number of errors between $0''\cdot 63$ and $0''\cdot 67$ was taken to represent the frequency through an error-range of $0''\cdot 05$, which must correspond in any mathematical formula to the independent variable $0''\cdot 65$. For those cases in which longer groups were employed, the actual number of observations was reduced so as to make it justly comparable with the number of observations in other parts of the series of groups: thus in the group extending from $-2''\cdot 38$ to $2''\cdot 13$, which extends over an error-range of $0''\cdot 25$, or five times the ordinary error-range, the actual number of observations was divided by 5 to make it comparable with the others, and the resulting quotient was held to correspond to the error or independent variable $-2''\cdot 30$.

It soon became evident that there was no marked discordance between the laws of distribution for positive and for negative values: and therefore the corresponding numbers were added together. Then, in order to remove small irregularities, the first number was added to the second, the second to the third, &c., and then the first sum was added to the second, the second to the third, &c., the first sum of the second order being held to correspond to the second original number; and so throughout. The extreme first and last were adopted without change. The numbers thus formed are evidently, on the whole, eight times as large as the original numbers. The numbers thus produced were laid down in a graphical representation, in which the abscissa was the magnitude of the "difference," or error of observation, and the ordinate was the corresponding number for eight times the frequency. Then a free curve was drawn by hand passing through the points representing these numbers. And this terminated the reference to the facts of observation. It appeared that, in the hand-drawn curve, the ordinate for error $=0$ might be taken as 124.

For a similar exhibition of the results of theory, or of the numbers given by the formula $A \times \epsilon^{-\frac{x^2}{c^2}}$, where $c =$ Modulus $= 1{\cdot}1973$, and where A evidently $= 124$, it was only necessary to calculate the expression,

$$\text{frequency} = 124 \times \epsilon^{\frac{-x^2}{(1\cdot1973)^2}},$$

or

$$\text{log. frequency} = 2{\cdot}0934217 - \frac{0{\cdot}4342945}{(1{\cdot}1973)^2} \times (\text{Error})^2.$$

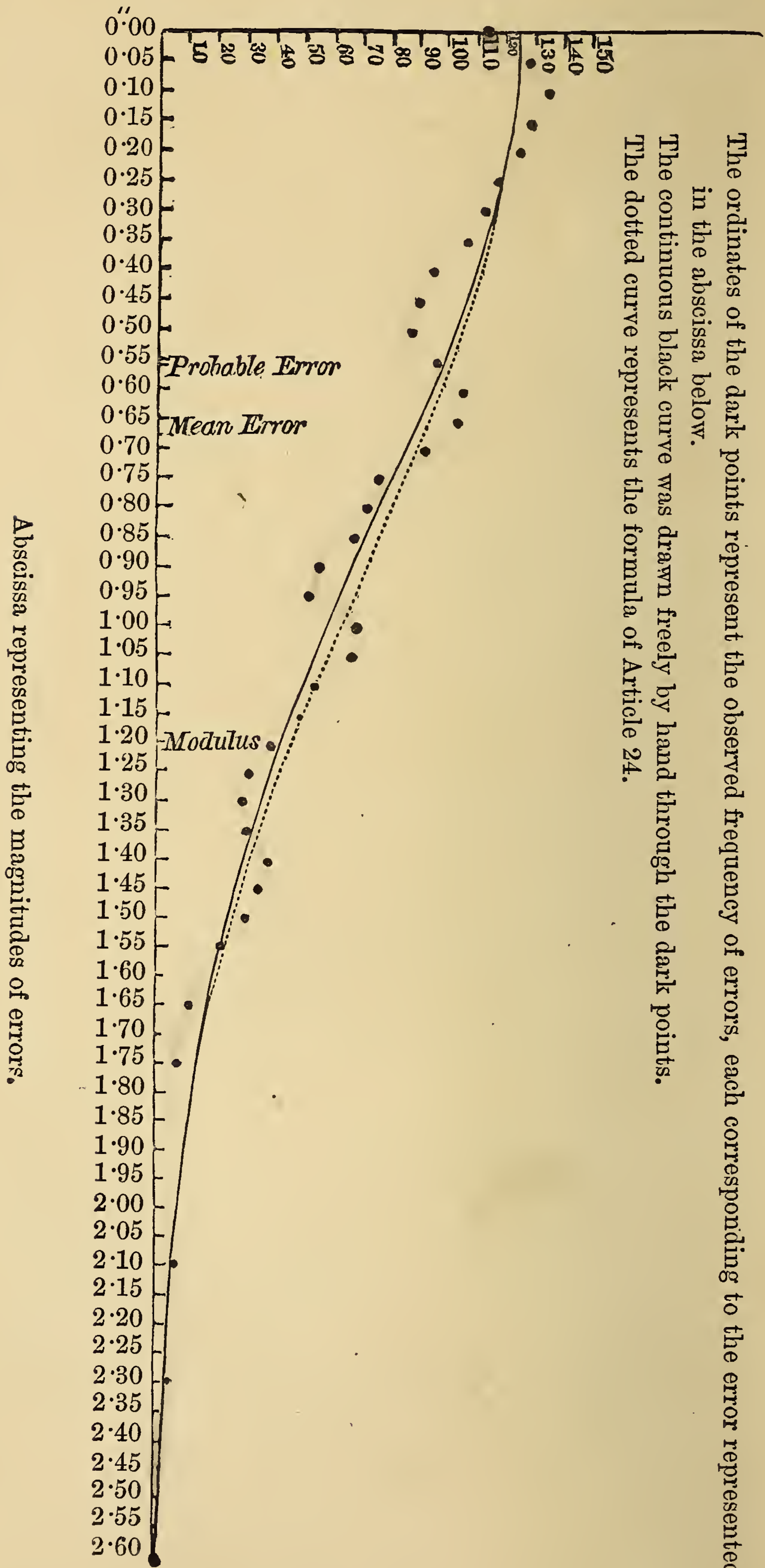
This calculation was made for every value of Error 0·05, 0·10, 0·15, &c. to 1·65, and then for 1·75, 2·10, 2·30, 2·50.

The comparison of the observed and the theoretical results is given in the following table.

Error of Observation.	Observed Frequency of Errors multiplied by the factor 8, corresponding to the error-range = 0″·05.	Ordinate of the Frequency-curve drawn by hand.	Theoretical value of Frequency.	Error of Observation.	Observed Frequency of Errors multiplied by the factor 8, corresponding to the error-range = 0″·05.	Ordinate of the Frequency-curve drawn by hand.	Theoretical value of Frequency.
0″·00	112	124	124·00	1″·30	29	35·5	38·14
0·05	127	124	123·78	1·35	33	32·5	34·78
0·10	135	123·5	123·14	1·40	38	30	31·60
0·15	129	122	122·07	1·45	35	27	28·61
0·20	125	120	120·59	1·50	30	24	25·81
0·25	116	118	118·71	1·55	22·3	22	23·21
0·30	114	116	116·45				
0·35	106	113·5	113·84	1·65	13·2	18	18·56
0·40	95	110	110·90				
0·45	90	105·5	107·67	1·75	8·4	14·5	14·64
0·50	88	102	104·16				
0·55	97	98	100·41				
0·60	105	93	96·46				
0·65	104	88·5	92·35				
0·70	91	84	88·10				
0·75	75	80	83·76				
0·80	72	76	79·35	2·10	6·4	6	5·73
0·85	69	71·5	74·91				
0·90	55	66·5	70·47				
0·95	52	62·5	66·07				
1·00	68	58	61·73	2·30	4·1	2·5	3·10
1·05	67	54	57.47				
1·10	53	50	53·32				
1·15	48	46	49·29				
1·20	39	42	45·41				
1·25	31	38·5	41·69				
				2·60	0·4	0·0	1·11

To exhibit more clearly to the eye the result of this comparison, the following diagram is prepared.

Ordinate representing the number of errors in each group ranging through 0·″05 of magnitude, multiplied by the factor 8.



It is evident that the formula represents with all practicable accuracy the observed Frequency of Errors, upon which all the applications of the Theory of Probabilities are founded: and the validity of every investigation in this Treatise is thereby established.

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